# Generalized Penrose Transform

## Quasi-$G$-equivariant $\mathcal{D}$–modules
## and
## Zuckerman Functor

Rémi LAMBERT

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# Remerciements

Le travail présenté dans cette thèse a suscité de nombreuses interrogations, un grand nombre de conversations intéressantes et instructives, parfois certains découragements mais surtout beaucoup de réjouissements. Cette thèse n'aurait certainement pas été accomplie sans la présence de certaines personnes et je tiens ici à leur exprimer toute ma gratitude.

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# Introduction

### Twistor theory and Penrose Transform

The twistor theory began in the late 1960's with the publication of some R. Penrose's papers [**68, 69, 70**] in which he introduced the notion of twistors and twistor geometry. The fundamental idea given by R. Penrose in his seminal paper entitled "*twistor algebra*" [**68**] is to replace the usual background spacetime used for the decription of many physical phenomena, by a new background space called "*twistor space*". The physical objects (spacetime, particles, fields, etc.) and the equations describing the phenomena are reinterpreted in this new twistor space, with the intent to get new insight into them. By analogy, R. Penrose's idea is similar to introducing the momentum space and using the Fourier transform to translate back and forth from spacetime to momentum space. The name "*twistor theory*" is derived from the Robinson congruence which is the natural realization of a twistor (see [**68**]).

There are several aspects to the twistor program initiated by R. Penrose. First, it successfully gave a new insight into many nonlinear classical field equations and even a new point of view for the classical linear field theories. New families of solutions to the nonlinear equations have been found and sometimes with the discovery of complete classification of the solutions. This approach was of interest to theoretical physicists to study solutions of equations in quantum field theory, general relativity theory and in some more recent attempts to quantize gravity. The mathematics of the twistor theory has the specific quality to use classical mathematical tools as well as others more modern. For example, this theory is based on some works from the 19$^{\text{th}}$ century by H.G. Grassmann, J. Plücker and F. Klein in projective and algebraic geometry and its formalism uses sheaf theory, sheaf cohomology invented by J. Leray in 1949 and some works of J.-P. Serre and H. Cartan whose ideas considerably helped to reinterpret classical phenomena in terms of the new language.

Using these twistor spaces, R. Penrose [**68**] showed in 1967 that the solutions of some important physical conformally invariant differential equations on $\mathbb{C}^4$ (massless fields in particular) could be expressed as contour integrals of holomorphic functions in twistor space. He also noticed that the functions appearing in these expressions were not unique, without however precisely determining the nature of the liberties they enjoyed. In 1981, M. EASTWOOD, R. PENROSE and R. O. WELLS solved this problem in **[26]** identifying this liberty as being the one of a representative of a Čech cohomology class. The resulting isomorphism between sheaf cohomology groups of the complex projective 3-space and solutions of conformal invariant equations over the complexified and compactified Minkowski spacetime has become known as the Penrose Transform.

Since the works of M. EASTWOOD, R. PENROSE and R. O. WELLS, the original Penrose transform has been generalized in several directions and studied in different contexts. Independently of R. PENROSE, W. SCHMIDT studied in 1967 the representations of Dolbeault cohomology groups using methods similar to the original Penrose transform (see **[76, 77]**). A generalization has been detailed by R. J. BASTON and M. G. EASTWOOD in **[7]** where the conformal group $\mathbb{C}^4$ is replaced by an arbitrary semisimple complex Lie group $G$. The generalized Penrose transform obtained in this way maps Dolbeault cohomology groups of homogeneous vector bundles on a complex homogeneous manifold $Z$ to solutions of invariant holomorphic differential equations on a complex submanifold of $Z$. Using the classical Lie group decomposition in terms of simple roots of the associated Lie algebra, Bott-Borel-Weil theorem and the Bernstein-Gelfand-Gelfand resolution, the authors describe an algorithmic way to compute explicitely this generalized Penrose transform.

Since solutions of differential equations on manifolds and cohomology classes are part of the bases of this generalized Penrose transform, the theory of sheaf cohomology and $\mathcal{D}$-modules was perfectly suited to it for its modern study. In 1996, A. D'AGNOLO and P. SCHAPIRA used this theory to generalize and study the R. PENROSE's correspondence in **[25]**. More recently, C. MARASTONI and T. TANISAKI in **[64]** studied the generalized Penrose transform between generalized flag manifolds over a complex algebraic group $G$, using M. KASHIWARA's correspondence between quasi-$G$-equivariant $\mathcal{D}_{G/H}$- modules and some kind of representation spaces which are $(\mathfrak{g}, H)$-modules (loosely, they are complex vector spaces endowed together with an action of the Lie algebra $\mathfrak{g}$ associated to $G$ and a action of $H$ which are compatible in some way) when $H$ is a closed algebraic subgroup of $G$. The subject of this thesis is inspired by this correspondence.

## Outline

The question studied in this thesis was introduced to us by J.-P. SCHNEIDERS. When $H$ is a closed algebraic subgroup of a given complex algebraic group $G$, M. KASHIWARA has established a correspondence of categories between (derived) category of quasi-$G$-equivariant $\mathcal{D}_{G/H}$-modules and (derived) category of $(\mathfrak{g}, H)$-modules. Since the R. PENROSE's integral transform was described and generalized on the quasi-equivariant $\mathcal{D}$-modules side by

C. Marastoni and T. Tanisaki in [**64**], an interesting problem was to determine the corresponding algebraic transform in terms of functors defined on some representation spaces via M. Kashiwara's equivalence and to explain a method to compute it.

To achieve this contribution, we proceed in three steps. Since the generalized Penrose transform for $\mathcal{D}$-modules is the composition of an inverse image functor and a direct image functor on the $\mathcal{D}$-module side, we first describe the algebraic analogs of these functors. More explicitely, if $H \subset K$ are two closed algebraic subgroups of a given complex algebraic group $G$, we show that the $\mathcal{D}$-module inverse image functor associated to the projection $g : G/H \to G/K$ is equivalent to the forgetful functor from the category of $(\mathfrak{g}, K)$-modules to the category of $(\mathfrak{g}, H)$-modules. We also show that the derived direct image functor $\mathbf{D}g_*$ on the $\mathcal{D}$-module side corresponds (up to a shift) to the algebraic Zuckerman functor $\mathrm{R}\Gamma_H^K$ which maps $(\mathfrak{g}, H)$-modules to $(\mathfrak{g}, K)$-modules (more precisely, objects of the derived categories).

The next step is to obtain an analog of the Bott-Borel-Weil theorem for computing the Zuckerman's functor image of "basic objects" of the category of $(\mathfrak{g}, H)$-modules. More precisely, we compute explicitely the image by the derived Zuckerman functor $\mathrm{R}\Gamma_H^K$ of generalized Verma modules $M_H(\lambda)$ associated to weights $\lambda$ which are integral for $\mathfrak{g}$ and dominant for the Lie subalgebra of $\mathfrak{g}$ corresponding to $H$. Those generalized Verma modules are the objects which correspond via M. Kashiwara's correspondence to the $\mathcal{D}$-modules which generate the Grothendieck group of the category of quasi-$G$-equivariant $\mathcal{D}_{G/H}$-modules of finite length.

Finally, we describe a method to analyse the image of such generalized Verma modules $M_H(\lambda)$ by the algebraic transform which corresponds to the so-called generalized Penrose transform. This analysis is inspired by the Bott-Borel-Weil theorem and Bernstein-Gelfand-Gelfand resolution used by R. J. Baston and M. G. Eastwood in [**7**].

## Content of the dissertation

The first chapter of this dissertation begins with some definitions and basic results about Lie algebra and Lie group representations. Some details about highest weight modules and their construction are given. Parabolic Lie subalgebras and parabolic Lie subgroups are defined with a particular rigour since they play a leading role in the main results. The $\mathscr{L}ie$ functor is also defined since the correspondence between Lie algebra representations and Lie group representations is widely applied in the entire dissertation. Since the core results consider generalized flag varieties, a section is devoted to the Kählerian manifolds and the link between compact simply connected homogeneous Kähler manifolds and the quotients of semisimple complex Lie groups by parabolic subgroups is given. We conclude the first chapter with a brief survey of the classical Bott-Borel-Weil theorem.

The second chapter deals with $\mathcal{D}$-modules and the generalization of the classical Penrose transform on generalized flag varieties. The (quasi-)equivariant framework is reminded and the fundamental operations such as tensor product, inverse image and direct image are defined. M. KASHIWARA's equivalence of categories between quasi-equivariant $\mathcal{D}$-modules on generalized flag manifolds and some representation spaces is recalled since this equivalence is central for the principal results which are given from chapter 3.

The third chapter contains the results of this thesis. We describe the algebraic analogs of the $\mathcal{D}$-modules operations of inverse and direct image by canonical projections between generalized flag manifolds with same underlying complex algebraic group. Then we prove in section 3.3 a Zuckerman version of Bott-Borel-Weil theorem. Finally, we deduce the algebraic analog to the generalized Penrose transform in theorem 3.4.1 and explain in section 3.4 an algorithmic way to analyse the effect produced by this algebraic Penrose transform.

# Introduction

### Théorie des Twisteurs et transformation de Penrose

La théorie des twisteurs débuta à la fin des années 1960 sous l'impulsion de certains articles de R. PENROSE [**68, 69, 70**] dans lesquels il introduisit la notion de twisteur et la géométrie des twisteurs. L'idée fondamentale proposée par R. PENROSE dans son article de référence intitulé « *twistor algebra* » [**68**] est de substituer à l'espace-temps – habituellement utilisé pour la description de nombreux phénomènes physiques – un autre espace qualifié d'« *espace des twisteurs* ». Les objets physiques (espace-temps, particules élémentaires, champs de forces, etc.) ainsi que les équations qui décrivaient ces phénomènes sont réinterprétés dans ce nouvel espace dans le but d'y apporter un nouveau regard. Par analogie, l'idée de R. PENROSE est similaire à l'introduction de l'espace des moments et à l'utilisation de la transformation de Fourier pour effectuer les traductions entre cet espace et l'espace-temps. Le nom de la « *théorie des twisteurs* » est dérivé de la congruence de Robinson qui est la réalisation naturelle d'un twisteur (voir [**68**]).

Le programme initié par R. PENROSE sous le nom de « *twistor program* » comporte plusieurs facettes. Historiquement, il a permis d'avoir un nouveau point de vue sur bon nombre d'équations non linéaires des champs ainsi qu'un regard novateur sur les classiques équations linéaires des champs. De nouvelles familles de solutions aux équations non linéaires ont été découvertes et parfois même la classification complète des solutions a été élucidée. De nombreux physiciens théoriciens ont utilisé cette approche dans l'étude des solutions d'équations en théorie quantique des champs, relativité générale et dans certaines tentatives récentes en gravitation quantique. Mathématiquement, la théorie des twisteurs a la particularité d'utiliser aussi bien des outils mathématiques classiques que d'autres très actuels. Cette théorie repose par exemple sur certains travaux du 19$^{e}$ siècle dûs à H.G GRASSMANN, J. PLÜCKER et F. KLEIN en géométrie projective et algébrique, et son formalisme contient de la théorie des faisceaux, de la cohomologie faisceautique inventée par J. LERAY en 1949 ainsi que quelques travaux de J.-P. SERRE et H. CARTAN dont les idées ont considérablement aidé à réinterpréter les phénomènes classiques en termes du nouveau langage.

En utilisant ces espaces de twisteurs, R. PENROSE [**68**] a établi en 1967 que les solutions de certaines équations différentielles invariantes conformes (les équations des champs de masse nulle en particulier) pouvaient être exprimées sous forme d'une intégrale curviligne de fonctions holomorphes dans cet espace de twisteurs. Il a également remarqué que les fonctions intervenant dans ces expressions n'étaient pas uniques, sans toutefois déterminer avec précision la nature des libertés dont elles jouissaient. C'est en 1981 que M. Eastwood, R. Penrose et R. O. Wells ont franchi cette étape dans [**26**] en identifiant cette liberté comme étant celle d'un représentant d'une classe de cohomologie de Čech. L'isomorphisme qui en résultait (par passage au quotient) entre des groupes de cohomologie de faisceaux modelés sur l'espace projectif de dimension 3 et des solutions d'équations différentielles conformément invariantes sur l'espace-temps de Minkowski compactifié et complexifié, est devenu la célèbre transformation de Penrose.

Depuis les travaux de M. Eastwood, R. Penrose et R. O. Wells, la transformation de Penrose a été généralisée dans plusieurs directions et étudiée dans différents contextes. En 1967, indépendamment de R. Penrose, W. Schmidt a étudié les représentations de groupes de cohomologie de Dolbeault en utilisant des méthodes similaires à celles de la transformation de Penrose (voir [**76, 77**]). Une généralisation a été détaillée par R. J. Baston et M. G. Eastwood dans [**7**], où le groupe conforme $\mathbb{C}^4$ est remplacé par un groupe de Lie semi-simple complexe quelconque. La transformation de Penrose généralisée obtenue de cette manière applique un groupe de cohomologie de Dolbeault de fibrés vectoriels homogènes modelés sur une variété homogène complexe $Z$, sur une solution d'équations différentielles holomorphes invariantes sur une sous-variété complexe de $Z$. En utilisant les décompositions classiques des groupes de Lie en terme de racines simples des algèbres de Lie associées, le théorème de Bott-Borel-Weil et la résolution de Bernstein-Gelfand-Gelfand, les auteurs décrivent un moyen algorithmique d'évaluer explicitement l'effet de cette transformation de Penrose généralisée.

Comme la notion de solution d'équations différentielles sur des variétés et celle de classes de cohomologie font partie des fondations de la transformation de Penrose généralisée, la théorie des faisceaux et des $\mathcal{D}$-modules est particulièrement adaptée pour son étude moderne. En 1996, A. D'Agnolo et P. Schapira ont utilisé ce langage pour généraliser et étudier la transformation de Penrose dans [**25**]. Plus récemment, C. Marastoni et T. Tanisaki ont étudié dans [**64**] la transformation de Penrose généralisée entre variétés de drapeaux généralisées sur un groupe algébrique complexe $G$ en employant la correspondance de catégories due à M. Kashiwara entre les $\mathcal{D}_{G/H}$-modules quasi-$G$-équivariants et certains espaces de représentations que sont les ($\mathfrak{g}$, $H$)-modules (qui, en simplifiant, sont des espaces vectoriels complexes munis à la fois d'une action de l'algèbre de Lie $\mathfrak{g}$ associée à $G$ et d'une action de $H$ satisfaisant à certaines conditions de compatibilité) lorsque $H$ est un sous-groupe algébrique fermé de $G$. Le sujet de cette thèse est inspiré de cette correspondance.

**Résumé**

Le problème étudié dans ce travail doctoral nous a été initialement posé par J.-P. Schneiders. Lorsque $H$ est un sous-groupe algébrique fermé d'un groupe algébrique complexe $G$ donné, M. Kashiwara a établi une équivalence de catégories entre la catégorie (dérivée) des $\mathcal{D}_{G/H}$-modules quasi-$G$-équivariants et la catégorie (dérivée) des $(\mathfrak{g}, H)$-modules. Étant donné que la transformation intégrale qui avait vu le jour grâce à R. Penrose avait été généralisée et décrite dans le contexte des $\mathcal{D}$-modules quasi-équivariants par C. Marastoni et T. Tanisaki dans **[64]**, un problème intéressant consistait à décrire l'analogue algébrique de cette transformation en terme de foncteurs définis sur des espaces de représentations via l'équivalence de M. Kashiwara et à donner une méthode pour la calculer explicitement.

Pour mener à bien cette contribution, nous procédons en trois étapes. Étant donné que la transformation de Penrose pour $\mathcal{D}$-modules est la composition d'un foncteur d'image inverse et d'un foncteur d'image directe, nous commençons par décrire les analogues algébriques de ces foncteurs. Plus explicitement, si $H \subset K$ sont deux sous-groupes algébriques fermés d'un groupe algébrique complexe $G$ donné, nous montrons que le foncteur d'image inverse au sens $\mathcal{D}$-modules associé à la projection $g : G/H \to G/K$ est équivalent au foncteur d'oubli de la catégorie des $(\mathfrak{g}, H)$-modules à valeurs dans la catégorie des $(\mathfrak{g}, K)$-modules. Nous montrons également que le foncteur dérivé d'image directe $\mathbf{D}g_*$ pour $\mathcal{D}$-modules correspond (à un décalage près) au foncteur de Zuckerman $\mathrm{R}\Gamma_H^K$ qui associe des $(\mathfrak{g}, K)$-modules aux $(\mathfrak{g}, H)$-modules (ou plus précisément, des objets des catégories dérivées).

L'étape suivante est l'obtention d'un analogue au théorème de Bott-Borel-Weil permettant d'évaluer l'image d'objets fondamentaux de la catégorie des $(\mathfrak{g}, H)$-modules par le foncteur de Zuckerman. Plus précisément, nous calculons explicitement l'image par le foncteur dérivé de Zuckerman $\mathrm{R}\Gamma_H^K$ des modules de Verma généralisés $M_H(\lambda)$ associés à un poids $\lambda$, qui est entier pour $\mathfrak{g}$ et dominant pour la sous-algèbre de Lie de $\mathfrak{g}$ correspondant à $H$. Ces modules de Verma généralisés sont les objets qui, via l'équivalence de catégories de M. Kashiwara, correspondent aux $\mathcal{D}$-modules engendrant le groupe de Grothendieck de la catégorie des $\mathcal{D}_{G/H}$-modules quasi-$G$-équivariants de longueur finie.

Finalement, nous décrivons une méthode pour analyser l'image de tels modules de Verma généralisés $M_H(\lambda)$ par la transformation algébrique correspondant à la transformation de Penrose généralisée. Inspirée des travaux de R. J. Baston et M. G. Eastwood dans **[7]**, cette analyse utilise le théorème de Bott-Borel-Weil et la résolution de Bernstein-Gelfand-Gelfand.

## Contenu de la dissertation

Le premier chapitre de cette dissertation doctorale débute par quelques définitions et résultats de base de la théorie des représentations d'algèbres et groupes de Lie. Quelques détails sur les modules de plus haut poids et leur construction sont donnés. Les sous-algèbres et sous-groupes paraboliques sont définis avec une attention particulière puisque leur structure a une importance capitale pour les résultats obtenus dans cette thèse. Le foncteur $\mathscr{L}ie$ est également défini puisque la correspondance entre les représentations des algèbres de Lie et celles de leurs groupes de Lie associés est largement utilisée. Puisque les résultats clés concernent les variétés de drapeaux généralisées, une section est consacrée aux variétés Kählériennes ainsi qu'au lien entre les variétés Kählériennes homogènes compactes simplement connexes et les quotients de groupes de Lie semi-simples complexes par leurs sous-groupes paraboliques. Nous concluons le premier chapitre par un bref rappel sur le théorème de Bott-Borel-Weil.

Le second chapitre est consacré aux $\mathcal{D}$-modules et à la généralisation de la transformation de Penrose classique aux variétés de drapeaux généralisées. Le contexte (quasi-)équivariant est rappelé et les opérations fondamentales telles que le produit tensoriel, l'image inverse et l'image directe y sont définies. L'équivalence de catégories due à M. Kashiwara entre les $\mathcal{D}$-modules quasi-équivariants sur des variétés de drapeaux généralisées et certains espaces de représentations est rappelée puis utilisée dans les résultats fondamentaux exposés à partir du chapitre 3.

Le troisième chapitre contient les résultats obtenus dans cette thèse. Nous décrivons les analogues algébriques aux opérations d'image inverse et d'image directe au sens $\mathcal{D}$-modules par des projections canoniques entre variétés de drapeaux généralisées construites sur le même groupe algébrique complexe. Ensuite, nous démontrons dans la section 3.3 une version du théorème de Bott-Borel-Weil pour le foncteur de Zuckerman. Finalement, nous en déduisons l'expression explicite de l'analogue algébrique à la transformation de Penrose généralisée dans le théorème 3.4.1 et nous détaillons dans la section 3.4 une méthode algorithmique pour analyser l'effet produit par cette transformation de Penrose algébrique.

CHAPTER 1

# Representation theory and generalized flag manifolds

This chapter recall the fundamental results about Lie algebras and Lie groups needed later. Its goal is double. First, we will introduce all the Lie algebras and Lie groups notations and results which will be useful in the sequel of this dissertation. Most of the material of this chapter is well-known and in such case, we will only point to useful references. Among all results presented here, some have proofs which remain quite difficult to find in litterature and are particularly interesting for the methods exposed later. Those proofs will be presented here. Secondly, we will give some reasons for hypotheses formulated in the next chapters which may sometimes appear as "restrictions". In particular, we will explain why we have to introduce some objects like parabolic subgroups to study compact simply connected Kählerian complex manifolds. Those manifolds are of main interest in theoretical physics, where Penrose transformation originally came into being. We will also give some details about the classification of parabolic subalgebras of a given semisimple complex Lie algebra.

We will devote an large part of this chapter to highest weight representations of Lie groups and Lie algebras. Lie group representations will have much importance in our results but since the language of Lie algebras is very comfortable for the classification of the representations, we will introduce the equivalence of categories between representations of Lie groups and representations of Lie algebras in terms of the $\mathscr{Lie}$ functor. Borel and parabolic subalgebras will be defined and some of their useful properties recalled. We will introduce the notions of weights which are integral or dominants for parabolic subalgebras and explain which weights are important in the classification of highest weight representations of parabolic subalgebras.

A section will be entirely devoted to Kählerian manifolds since most of the results obtained in this dissertation concern some compact Kählerian manifolds which are generalized flag manifolds. This chapter will end with a brief survey of the classical Bott-Borel-Weil theorem, since an analog of this result will be used for $\mathcal{O}$-modules and an algebraic version will be obtained in chapter 2 in terms of the Zuckerman functor and generalized Verma modules.

We will not define here what a Lie algebra or a Lie group are, for instance. Some basic definitions and results will be simply recalled and the reader who will be interested in some additional background information will find them in the main reference books used in this chapter: [**19, 28, 35, 42, 44, 55**] and [**87**]. Some additional interesting informations can be found in [**8, 13, 41, 54, 76, 77, 81, 85**].

Every Lie algebra is intended to be over $\mathbb{C}$ and finite dimensional.

## 1.1. Lie algebras and representation theory survey

Suppose that $\mathfrak{g}$ is a finite dimensional complex Lie algebra, and denote by $[\cdot, \cdot]$ the Lie bracket on $\mathfrak{g}$.

For any element $g$ of $\mathfrak{g}$, we denote by $\mathrm{ad}(g)$ the adjoint endomorphism of $\mathfrak{g}$ defined by

$$\mathrm{ad}(g) \ : \ \mathfrak{g} \to \mathfrak{g} \ : \ g' \mapsto [g, g'].$$

It follows from basic properties of the Lie bracket that $\mathrm{ad}(g)$ is a derivation of $\mathfrak{g}$ (i.e. satisfies the Leibniz rule) for every $g \in \mathfrak{g}$.

An *ideal in* $\mathfrak{g}$ is a complex vector subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[g, h] \in \mathfrak{h}$ for any $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$. This definition is the natural analog of an ideal in an associative algebra. However, since the commutator is skew-symmetric, there is no difference between left and right ideals. The Lie algebra $\mathfrak{g}$ is *simple* if $\dim^{\mathbb{C}} \mathfrak{g} > 1$ and $\mathfrak{g}$ has no proper ideals.

We say that $\mathfrak{g}$ is *abelian* (or *commutative*) if $[\mathfrak{g}, \mathfrak{g}] = \{0\}$. Of course, every finite dimensional complex vector space may be viewed as an abelian finite dimensional complex Lie algebra by defining the Lie bracket always equal to zero. The abelian Lie algebras are interesting examples since the Lie theory is easy in that context: every irreducible representation of such Lie algebras is 1-dimensional, for instance.

Define a sequence of ideals $\mathfrak{g}^{(i)}$ in $\mathfrak{g}$, called *derived series of* $\mathfrak{g}$, by setting $\mathfrak{g}^{(0)} = \mathfrak{g}$ and

$$\mathfrak{g}^{(i+1)} = \left[\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}\right], \ i \in \mathbb{N}\backslash\{0\}.$$

We call $\mathfrak{g}$ *solvable* if $\mathfrak{g}^{(n)} = \{0\}$ for some $n \in \mathbb{N}$. For instance, abelian Lie algebras are solvable. One can show the following classical result:

PROPOSITION 1.1.1. *The Lie algebra* $\mathfrak{g}$ *is solvable if and only if there exists a sequence of subalgebras* $\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \ldots \supset g_p = \{0\}$ *such that for every* $0 \leq i \leq p-1$, $\mathfrak{g}_{i+1}$ *is an ideal in* $\mathfrak{g}_i$ *and the quotient* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

PROOF. See [**55**] proposition 1.23 p.40. □

The sum of two solvable ideals is again a solvable ideal (see [**55**] §1.2 or [**42**] p.11). By induction, we obtain that any finite sum of solvable ideals is also solvable. Finite dimensionality of $\mathfrak{g}$ implies that the sum of all solvable ideals in $\mathfrak{g}$ is in fact a finite sum and hence is a solvable ideal in $\mathfrak{g}$. Let $\mathrm{rad}\,\mathfrak{g}$ to be this sum, which is obviously the unique maximal solvable ideal in $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is *semisimple* if $\mathrm{rad}\,\mathfrak{g} = 0$. This condition is equivalent to the fact that there is no nonzero solvable ideals in $\mathfrak{g}$. Moreover, any simple Lie alebra is semisimple.

PROPOSITION 1.1.2. *For any Lie algebra $\mathfrak{g}$, the quotient $\mathfrak{g}/\,rad\,\mathfrak{g}$ is semisimple. Conversely, if $\mathfrak{h}$ is a solvable ideal in $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{h}$ is semisimple, then $\mathfrak{h} = rad\,\mathfrak{g}$.*

PROOF. See **[55]** proposition 1.14 p.33. □

This shows that any Lie algebra can be included in a short exact sequence

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}' \to 0$$

where $\mathfrak{h}$ is solvable and $\mathfrak{g}'$ semisimple. In fact, we have a stronger result:

THEOREM 1.1.3 (LEVI). *Any Lie algebra can be written as a direct sum*

$$\mathfrak{g} = rad\,\mathfrak{g} \oplus \mathfrak{g}'$$

*where $\mathfrak{g}'$ is a semisimple subalgebra of $\mathfrak{g}$. Such a decomposition is called a Levi decomposition of $\mathfrak{g}$.*

The proof of this result essentially reduces to show that cohomology $H^2(\mathfrak{g}, \mathbb{C})$ vanishes for semisimple $\mathfrak{g}$ and may be found in **[44]** or **[55]**.

Although the radical is uniquely determined by $\mathfrak{g}$, the Levi decomposition is in general not unique. In fact, Anatolii I. MALCEV (see **[63]**), Morikuni GOTÔ (see **[31]**) and Harish-Chandra (see **[36]**) showed that if there are two Levi decompositions $\mathfrak{g} = \mathrm{rad}\,\mathfrak{g} \oplus \mathfrak{g}'_1 = \mathrm{rad}\,\mathfrak{g} \oplus \mathfrak{g}'_2$ with $\mathfrak{g}'_1$ and $\mathfrak{g}'_2$ semisimple subalgebras of $\mathfrak{g}$, then there is an automorphism in $\mathfrak{g}$ which carries $\mathfrak{g}'_1$ onto $\mathfrak{g}'_2$. Thus the Levi decomposition is unique only in the trivial case when the semisimple part of $\mathfrak{g}$ is an ideal in $\mathfrak{g}$, and the study of the structure of a Lie algebra is almost reduced to studying semisimple and solvable Lie algebras.

Another notion which be important in the sequel is the nilpotency and is usually defined similarly to solvable Lie algebras. Define a sequence of ideals $D_i\mathfrak{g}$ in $\mathfrak{g}$ (called *lower central series*) by setting $D_0\mathfrak{g} = \mathfrak{g}$ and

$$D_{i+1}\mathfrak{g} = [\mathfrak{g},\, D_i\mathfrak{g}],\;\; i \in \mathbb{N}\backslash\{0\}.$$

We can show a similar result to proposition 1.1.1:

PROPOSITION 1.1.4. *We have $D_n\mathfrak{g} = \{0\}$ for large enough integer $n$ if and only if there is a sequence of ideals $\mathfrak{g}_0 = \mathfrak{g} \supset \mathfrak{g}_1 \supset \ldots \supset g_p = \{0\}$ such that $[\mathfrak{g},\, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ for every integer $i$.*

One direction of the proof consists to define $g_i = D_i\mathfrak{g}$ for every $i \in \mathbb{N}$, and the opposite direction is obtained by showing that $D_i\mathfrak{g} \subset \mathfrak{g}_i$ by induction.

If there is an integer $n$ such that $D_n\mathfrak{g} = \{0\}$, then $\mathfrak{g}$ is called *nilpotent*. Of course, abelian algebras are nilpotent.

EXAMPLE 1.1.5. If $\mathfrak{g}$ is the Lie algebra of square matrices with complex coefficients, then the subalgebra of upper triangular matrices is solvable and the subalgebra of all strictly upper triangular matrices is nilpotent.

Let $\mathfrak{z}(\mathfrak{g})$ be the center of $\mathfrak{g}$. In other words, we have

$$\mathfrak{z}(\mathfrak{g}) = \{g \in \mathfrak{g} \ : \ [g, g'] = 0 \text{ for every } g' \in \mathfrak{g}\}.$$

Obviously, the center of $\mathfrak{g}$ is a solvable ideal. The Lie algebra $\mathfrak{g}$ is called *reductive* if $\operatorname{rad}\mathfrak{g} = \mathfrak{z}(\mathfrak{g})$ ie. if $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple. Examples of reductive Lie algebras are all abelian algebras, semisimple Lie algebras, and the Lie algebra of $n \times n$ matrices with complex coefficients.

A *Cartan subalgebra* of $\mathfrak{g}$ is a nilpotent subalgebra of $\mathfrak{g}$ which equals its normalizer in $\mathfrak{g}$. Explicitely, a nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a Cartan subalgebra if

$$\mathfrak{h} = \{g \in \mathfrak{g} \ : \ [g, \mathfrak{h}] \subset \mathfrak{h}\}.$$

EXAMPLE 1.1.6. If $\mathfrak{g}$ is the Lie algebra of square matrices with complex coefficients, its subspace of diagonal matrices with trace zero is a Cartan subalgebra.

It is a classical fact that Cartan subalgebras exists and we can show that all Cartan subalgebras of $\mathfrak{g}$ are mutually conjugated (see Chevalley's theorem [**87**] p.263 or [**42**] p.82). The dimension of a Cartan subalgebra of $\mathfrak{g}$ is called the *rank of* $\mathfrak{g}$.

Moreover, one can show that every Cartan subalgebra of $\mathfrak{g}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$. However, a maximal nilpotent subalgebra of $\mathfrak{g}$ is not necessarily a Cartan subalgebra. For instance, consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{h}$ being the Lie subalgebra of $\mathfrak{g}$ which are strictly upper triangular. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$ but its normalizer is the subalgebra of $\mathfrak{g}$ constituted by upper triangular matrices. Hence in this example, the subalgebra $\mathfrak{h}$ is not a Cartan subalgebra of $\mathfrak{g}$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{h}^* = \operatorname{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ its dual. For $\alpha \in \mathfrak{h}^*$, define

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \ : \ \operatorname{ad}(h)(g) = \alpha(h)g \text{ for all } h \in \mathfrak{h}\}.$$

DEFINITION 1.1.7. An element $\alpha \in \mathfrak{h}^*$ is a *root* of $(\mathfrak{g}, \mathfrak{h})$ if $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$. Denote by $\Delta$ the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. The space $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ is the *root subspace associated to* $\alpha$.

The set $\Delta$ is finite and we have the following decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{1}$$

The equality (1) is the *root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

The Jacobi identity on $\mathfrak{g}$ implies that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$ is a subset of $\mathfrak{g}_{\alpha+\beta}$ (which can be trivial).

Let $n$ be the rank of $\mathfrak{g}$ and $h_1$, ..., $h_n$ be a basis of $\mathfrak{h}$. When $\varphi \in \mathfrak{h}^*$, we say that $\varphi$ is *positive* (resp. *negative*) and we write $\varphi \succ 0$ (resp. $\varphi \prec 0$) if there exists an index $k$ such that $\varphi(h_i)$ for $1 \leq i \leq k-1$ and $\varphi(h_k) > 0$ (resp. $\varphi(h_k) < 0$). When $\varphi$, $\psi \in \mathfrak{h}^*$, we also say that $\varphi \succ \psi$ if $\varphi - \psi \succ 0$. One can show that it is a linear and total order on $\mathfrak{h}^*$, which is called the *lexicographic ordering on* $\mathfrak{h}^*$. The set of positive (resp. negative) roots is denoted by $\Delta^+$ (resp. $\Delta^-$).

We say that a root $\alpha$ is *simple* if $\alpha \succ 0$ and if $\alpha$ does not decompose as a sum of two positive roots. The set of simple roots is denoted by $\mathcal{S}$. We obtain an ordering of simple roots by $\alpha_1 \prec \alpha_2 \prec \ldots \prec \alpha_n$. The set $\{1, 2, \ldots, n\}$ will be denoted by $I_0$.

We have the following:

PROPOSITION 1.1.8. *If $\lambda$ is a root and is decomposed by*

$$\lambda = \sum_{j=1}^{n} c_j \, \alpha_j,$$

*then all nonzero $c_j$ have the same sign and all $c_j$ $(1 \leq j \leq n)$ are integers.*

PROOF. See **[55]**, proposition 2.49 p.155. □

The Lie algebra $\mathfrak{g}$ admits a particular bilinear form, defined as follows.

DEFINITION 1.1.9. For $u, v \in \mathfrak{g}$, the trace of the composition $\mathrm{ad}(u) \circ \mathrm{ad}(v)$ is a bilinear form on $\mathfrak{g}$, called the *Killing form* and is denoted by $\beta$.

The Killing form is symmetric, associative and ad-invariant in the sense that

$$\beta([x, y], z) = \beta(x, [y, z])$$

for any $x$, $y$, $z \in \mathfrak{g}$. Moreover, since $\mathfrak{g}$ is supposed to be semisimple, the restriction of the Killing form on $\mathfrak{h}$ is non singular. Since the canonical application

$$\mathfrak{h} \to \mathfrak{h}^* \; : \; x \mapsto (y \in \mathfrak{h} \mapsto \beta(x, y)) \tag{2}$$

defines an isomorphism between $\mathfrak{h}$ and $\mathfrak{h}^*$, we obtain a non-degenerate bilinear form on $\mathfrak{h}^*$ which will be denoted by $(\cdot, \cdot)$. When $(\cdot, \cdot)$ is restricted to $\mathfrak{h}^*_{\mathbb{R}}$, where

$$\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \ : \ \alpha_i(h) \in \mathbb{R} \text{ for all } i \in I_0\},$$

it gets real valued, symmetric and positive definite. For $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, let

$$|\lambda|^2 = (\lambda, \lambda).$$

For $\alpha \in \Delta$, we denote by $\alpha^\vee \in \mathfrak{h}^*$ the corresponding coroot defined by

$$\alpha^\vee = 2\alpha/(\alpha, \alpha).$$

The *Cartan integers*

$$c_{i,j} = (\alpha_i, \alpha_j^\vee) \quad (i, j \in I_0)$$

encode the totality of the structure of the lie algebra $\mathfrak{g}$ (see **[87]**) since the knowledge of the Cartan integers suffices to reconstruct the Lie algebra $\mathfrak{g}$ (up to isomorphism). A method to achieve this was given by J.P. Serre in **[82]** p.52, where he considers Cartan matrices, ie. square matrices $A = (a_{i,j})_{1\leq i, j\leq k}$ which satisfies simultaneously the following conditions:

(1) $a_{i,j} \in \{-3, -2, -1, 0, 1, 2\}$ for every $1 \leq i, j \leq k$;
(2) $a_{i,i} = 2$ for every $1 \leq i \leq k$;
(3) $a_{i,j} \leq 0$ for every $1 \leq i \neq j \leq k$;
(4) there is a diagonal matrix $D$ such that $DAD^{-1}$ is symmetric and positive definite.

The semisimple complex Lie algebra $\mathfrak{g}$ which corresponds up to isomorphism to a given Cartan matrix can be reconstructed by $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{e} \oplus \mathfrak{f}$ where $\mathfrak{h}$, $\mathfrak{e}$ and $\mathfrak{f}$ are respectively generated by the sets $\{h_1, \ldots, h_k\}$, $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_k\}$) with elements which satisfies the Chevalley-Serre relations:

(1) $[h_i, h_j] = 0$;
(2) $[e_i, f_i] = h_i$ and $[e_i, f_j] = 0$ when $i \neq j$;
(3) $[h_i, e_j] = a_{i,j}e_j$;
(4) $[h_i, f_j] = -a_{i,j}f_j$;
(5) $(\mathrm{ad}(e_i))^{1-a_{i,j}}(e_j) = 0$;
(6) $(\mathrm{ad}(f_i))^{1-a_{i,j}}(f_j) = 0$

for every $1 \leq i, j \leq k$. Moreover, the Lie algebra $\mathfrak{g}$ has rank $k$ and the $h_i$ $(1 \leq i \leq k)$ generate a Cartan subalgebra.

Let $\lambda_i$ $(i \in I_0)$ be the elements of $\mathfrak{h}^*$ which are defined dually to the co-roots $\alpha_i^\vee$ $(i \in I_0)$ by the system

$$(\lambda_i, \alpha_j^\vee) = \delta_{i,j} \quad (i, j \in I_0).$$

Obviously, the set $\{\lambda_i \ : \ i \in I_0\}$ is a basis for $\mathfrak{h}^*$ and any element $\lambda$ of $\mathfrak{h}^*$ can be decomposed as

$$\lambda = \sum_{i\in I_0} (\lambda, \alpha_i^\vee)\, \lambda_i. \tag{3}$$

This decomposition is called the *dual decomposition* of $\lambda$ and will be useful to define the lattice of integral weights of $\mathfrak{g}$ (see p.13).

For each $\alpha \in \Delta$, we define the *$\alpha$-wall* $W_\alpha$ to be the hyperplane in $\mathfrak{h}_\mathbb{R}^*$ perpendicular to $\alpha$, i.e.

$$W_\alpha = \{\beta \in \mathfrak{h}_\mathbb{R}^* \ : \ (\beta, \alpha) = 0\}.$$

The Weyl chambers are the connected components of $\mathfrak{h}_\mathbb{R}^* \backslash \bigcup_{\alpha \in \Delta} W_\alpha$. Among all Weyl chambers, there is a particular which is called *fundamental Weyl chamber*, whose elements $\alpha$ are characterized by $(\alpha, \alpha_i) > 0$ for all $i \in I$.

We now introduce some actions on $\mathfrak{h}_\mathbb{R}^*$ which will generate the Weyl group.

DEFINITION 1.1.10. For $\alpha \in \mathfrak{h}_\mathbb{R}^* \backslash \{0\}$, we define $\sigma_\alpha$ to be the reflection perpendicular to $W_\alpha$. Explicitely, we define:

$$\sigma_\alpha \ : \ \mathfrak{h}_\mathbb{R}^* \to \mathfrak{h}_\mathbb{R}^* \ : \ \lambda \mapsto \lambda - (\lambda, \alpha^\vee)\, \alpha.$$

The reflections associated to simple roots are called *simple reflections.*

We can easily verify the following properties of reflections:

PROPOSITION 1.1.11. *Let $\sigma_\alpha$ be the reflection associated to $\alpha \in \mathfrak{h}_\mathbb{R}^* \backslash \{0\}$.*

1. $\sigma_\alpha(\alpha) = -\alpha$*;*
2. *we have $\sigma_\alpha = \sigma_{-\alpha}$ and $\sigma_\alpha \circ \sigma_\alpha$ is the identity on $\mathfrak{h}_\mathbb{R}^*$;*
3. *we have $(\sigma_\alpha \lambda, \mu) = (\lambda, \sigma_\alpha \mu)$ for all $\lambda, \mu \in \mathfrak{h}_\mathbb{R}^*$, and the scalar product $(\cdot, \cdot)$ is invariant under reflections;*
4. $\sigma_\alpha W_\alpha = W_\alpha$.

Geometrically, reflections are invertible linear transformations leaving pointwise fixed some wall and sending any element orthogonal to that wall into its negative.

PROPOSITION 1.1.12. *If $\lambda \in \Delta$, the restriction of $\sigma_\lambda$ to $\Delta$ is a permutation of $\Delta$.*

PROOF. See [**87**], lemma 4.3.14 p.279. □

DEFINITION 1.1.13. The *Weyl group of* $\mathfrak{g}$ is the group $W_\mathfrak{g}$ generated by the reflections $\sigma_\alpha$ $(\alpha \in \Delta)$.

A direct and useful consequence of proposition 1.1.11 is that

$$(w\lambda, \mu) = (\lambda, w^{-1}\mu)$$

for every $w \in W_\mathfrak{g}$ and $\lambda, \mu \in \mathfrak{h}_\mathbb{R}^*$.

The Weyl group has many properties (see [**87**]). For instance, one can prove (see [**42**] p.51) that the Weyl group of $\mathfrak{g}$ is generated by the identity and the reflections associated to simple roots of $\mathfrak{g}$. An additional fundamental property for our interest is the following one.

PROPOSITION 1.1.14. *The Weyl group of $\mathfrak{g}$ acts simply transitively on the set of Weyl chambers. Explicitely, for any two $C_1$, $C_2$ Weyl chambers, there exists precisely one $w$ in $W_{\mathfrak{g}}$ such as $wC_1 = C_2$.*

PROOF. See [**87**], theorem 4.3.18 p.282. □

We can deduce from the latter result that for every $\lambda \in \Delta$ which is not in a wall, there exists a unique $w \in W_{\mathfrak{g}}$ such as $(w\lambda, \alpha) > 0$ for all $\alpha \in \Delta^+$. Since this observation will be recurrently used in chapters 2 and 3, we define a notion of singularity as follows:

DEFINITION 1.1.15. An element $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is *singular for* $\mathfrak{g}$ if it is on some wall of $\mathfrak{g}$.

It turns out, as a consequence of Harish Chandra's theorem about characters (see [**42**] p.130), that the most important action of the Weyl group on weights is not the straightforward one obtained by composition of reflections but rather an affine action. Before its definition, let us denote by $\delta$ the Weyl vector of $\mathfrak{g}$. Explicitely, we set

$$\delta = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$$

We check that if $\alpha$ is a simple root of $\mathfrak{g}$, then $\sigma_\alpha \delta = \delta - \alpha$ since simple reflexions permute simple roots. Here is now the definition of the affine action of the Weyl group.

DEFINITION 1.1.16. If $w \in W_{\mathfrak{g}}$ and $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, we denote by

$$w \cdot \lambda = w(\lambda + \delta) - \delta$$

the *affine* action of $w$ on $\lambda$.

This affine action allows to determine the highest weight of the resulting representation while calculating cohomologies via the Bott-Borel-Weil theorem (see theorem 1.6.1).

A crucial distinction occurs in the Bott-Borel-Weil theorem 1.6.1, depending on whether $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ has a non-trivial stabilizer under the affine action of the Weyl group. For this purpose, we introduce the following definition.

DEFINITION 1.1.17. An element $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ is *affinely singular for* $\mathfrak{g}$ if it has a non-trivial stabilizer under the affine action of the Weyl group.

We can see that if $\lambda$ is affinely singular, then $\lambda + \delta$ is singular and there is some $w \in W_{\mathfrak{g}}$ and $i \in I_0$ such that $(w(\lambda + \delta), \alpha_i) = 0$. Of course, if $\lambda + \delta$ is singular, then $\lambda$ is affinely singular and this fact allows to detect the affine singularity of $\lambda$ by looking at the orbit of $\lambda + \delta$ under the Weyl group action.

There is a length function which can be defined on the Weyl group $W_{\mathfrak{g}}$.

DEFINITION 1.1.18. When $w \in W_{\mathfrak{g}} \backslash \{\mathrm{id}\}$ is decomposed as $\sigma_{\tau(1)} \circ \ldots \circ \sigma_{\tau(p)}$ where $\tau(1)$, ..., $\tau(p)$ are simple roots of $\mathfrak{g}$, we call this expression *reduced* if $p$ is minimum, and in such case, we define $\ell(w) = p$ to be the *length* of $w$. Of course, we define $\ell(\mathrm{id}) = 0$.

We can show (see [**42**], lemma A p.52) that the length of $w \in W_{\mathfrak{g}}$ is the number of elements in $w^{-1}\Delta^- \cap \Delta^+$.

There is a unique element of the Weyl group which maps the set of positive roots to the set of negative roots and we denote it by $w_{\mathfrak{g}}$. This element is in fact the longest element of the Weyl group, i.e. we have $\ell(w) < \ell(w_{\mathfrak{g}})$ for every $w \in W_{\mathfrak{g}} \backslash \{w_{\mathfrak{g}}\}$. Of course, the definition of $w_{\mathfrak{g}}$ gives that $\ell(w_{\mathfrak{g}}) = \#\Delta^+$. This equality will be generalized in lemma 1.3.6.

## 1.2. Lie algebras and Lie groups correspondence: the $\mathscr{L}ie$ functor

Some proofs of the most fundamental and well-known results will not appear in this section but can be read in [**19, 41, 54, 55**] or [**87**].

Denote by **LieG** the category of complex holomorphic Lie groups and by **LieA** the category of complex Lie algebras.

Let $G$ be a complex holomorphic Lie group. Denote by $L$ the left action of $G$ on $G$ defined by $L_g(g') = gg'$ for $g$, $g' \in G$. The set of left-invariant holomorphic vector fields on $G$ is a Lie algebra over $\mathbb{C}$, denoted by $\mathscr{L}ie(G)$ and called the *Lie algebra associated to* $G$. Since the map which associates to $X \in \mathscr{L}ie(G)$ the tangent vector field defined by $X$ at the identity of $G$ is an isomorphism, we can identify $\mathscr{L}ie(G)$ to the tangent space of $G$ at the identity.

Moreover, if $f : G_1 \to G_2$ is a morphism of complex Lie groups, we get a morphism of Lie algebras $\mathscr{L}ie(f) : \mathscr{L}ie(G_1) \to \mathscr{L}ie(G_2)$ by differentiation at the identity. Hence $\mathscr{L}ie$ is a covariant functor from **LieG** to **LieA**.

We have the following elementary result:

PROPOSITION 1.2.1. *If $f : G_1 \to G_2$ is a morphism of Lie groups, the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{\quad f \quad} & G_2 \\ {\scriptstyle \exp}\uparrow & & \uparrow{\scriptstyle \exp} \\ \mathscr{L}ie(G_1) & \xrightarrow[\mathscr{L}ie(f)]{} & \mathscr{L}ie(G_2) \end{array}$$

*commutes, where* exp *denotes the exponential map.*

The connected component of the identity of a complex Lie group $G$ is generated by $\exp(\mathscr{L}ie(G))$. Consequently, morphisms of Lie groups are determined on the connected component of the identity by their differential at the identity. In other words, the restriction of the functor $\mathscr{L}ie$ to the full subcategory of **LieG** consisting of connected Lie groups is faithful.

Moreover, if $\alpha$ is a Lie algebra morphism between $\mathscr{L}ie(G_1)$ and $\mathscr{L}ie(G_2)$ where $G_1, G_2 \in \mathbf{LieG}$, the proposition 1.2.1 suggests a method to lift $\alpha$ to a morphism $f$ such that $\mathscr{L}ie(f) = \alpha$. Since the exponential mapping is a local diffeomorphism between a neighborhood of 0 in $\mathscr{L}ie(G_1)$ and a neighborhood of the identity in $G_1$, we can define $f$ in the connected component of $G_1$ which contains the identity. We have the following:

PROPOSITION 1.2.2. *Let $G_1$, $G_2$ be two complex Lie groups. If $G_1$ is simply connected then for every Lie algebra morphism $\alpha : \mathscr{L}ie(G_1) \to \mathscr{L}ie(G_2)$, there exists a unique Lie group morphism $f : G_1 \to G_2$ such that $\mathscr{L}ie(f) = \alpha$.*

Hence the restriction of the functor $\mathscr{L}ie$ to the full subcategory of **LieG** consisting of simply connected Lie groups is not only faithful but fully faithful.

It is possible to construct an inverse functor to $\mathscr{L}ie$ when one restricts to simply connected Lie groups.

PROPOSITION 1.2.3. *Let $\mathfrak{g}_1$ be a complex Lie algebra. There exists a pair $(G(\mathfrak{g}_1), i_{\mathfrak{g}_1})$ where $G(\mathfrak{g}_1)$ is a simply connected Lie group over $\mathbb{C}$ and $i_{\mathfrak{g}_1} : \mathfrak{g}_1 \to \mathscr{L}ie(G(\mathfrak{g}_1))$ is an isomorphism of Lie algebras, such that for every morphism of Lie algebras $f : \mathfrak{g}_1 \to \mathscr{L}ie(G_2)$ where $G_2$ is a complex Lie group, there is a unique morphism of Lie groups $\overline{f} : G(\mathfrak{g}_1) \to G_2$ such that the diagram*

$$\begin{array}{ccc} \mathfrak{g}_1 & \xrightarrow{\quad f \quad} & \mathscr{L}ie(G_2) \\ {\scriptstyle i_{\mathfrak{g}_1}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathscr{L}ie(G(\mathfrak{g}_1)) & \xrightarrow[\mathscr{L}ie(\overline{f})]{} & \mathscr{L}ie(G_2) \end{array}$$

*commutes. In particular, such pairs are unique up to isomorphism and the mapping $\mathfrak{g} \mapsto G(\mathfrak{g})$ defines a functor from the category of complex Lie algebras to the category of simply connected complex Lie groups.*

The essential surjectivity argument of this result is the third fundamental theorem of Lie which states that every Lie algebra is isomorphic to the Lie algebra of a Lie group.

Thus we have a quasi inverse functor of $\mathscr{L}ie$ when restricted to the full subcategory of **LieG** whose objects are simply connected and we have the following equivalence of categories:

PROPOSITION 1.2.4. *The functor $\mathscr{L}ie$ induces an equivalence of categories from the category of simply connected Lie groups over $\mathbb{C}$ to the category of finite dimensional complex Lie algebras. Its quasi inverse functor is given by $\mathfrak{g} \mapsto G(\mathfrak{g})$.*

A Lie group is said to be *reductive*, *solvable*, *nilpotent* or *semisimple* if it is connected and if its associated Lie algebra has the same property.

**1.2.1. Lie algebra and Lie group representations.** This section deals mainly with finite dimensional representations of semisimple complex Lie algebras. The references used for this topic are [**42, 44, 54, 55**] and [**87**].

DEFINITION 1.2.5. A *finite dimensional representation* of $\mathfrak{g}$ is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ where $V$ is a finite dimensional complex vector space and $\mathfrak{gl}(V)$ is endowed with the commutator bracket.

EXAMPLE 1.2.6. The adjoint representation of $\mathfrak{g}$ is defined by

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : g \mapsto \left(g' \mapsto [g, g']\right).$$

We can now introduce the notion of module over a Lie algebra.

DEFINITION 1.2.7. A complex vector space $V$ endowed with a $\mathbb{C}$-bilinear operation $\mathfrak{g} \otimes_{\mathbb{C}} V \to V$ denoted by $(g, v) \mapsto g \cdot v$ (or simply $gv$) is called a $\mathfrak{g}$-*module* if

$$[g, g'] \cdot v = g \cdot (g' \cdot v) - g' \cdot (g \cdot v)$$

for every $g, g' \in \mathfrak{g}$, $v \in V$.

If $\rho$ is a Lie algebra representation of $\mathfrak{g}$ on $V$, then $V$ can be viewed as a $\mathfrak{g}$-module using the action $\mathfrak{g} \otimes_{\mathbb{C}} V \to V$ defined by

$$g \cdot v = \rho(g)(v), \; g \in \mathfrak{g}, \, v \in V.$$

Conversely, if $V$ is a $\mathfrak{g}$-module, it can be seen as a representation space of the Lie algebra $\mathfrak{g}$ since the morphism

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) : g \mapsto (v \mapsto g \cdot v)$$

is a Lie algebra homomorphism. Thus there is a correspondence between $\mathfrak{g}$-modules and representations of $\mathfrak{g}$. Moreover, there is a correspondence between representations of $\mathfrak{g}$ and modules over $\mathcal{U}(\mathfrak{g})$.

DEFINITION 1.2.8. A *homomorphism of* $\mathfrak{g}$-*modules* (or $\mathfrak{g}$-*module map*) is a linear map which preserves the structure of $\mathfrak{g}$-module. More precisely, if $V$ and $W$ are two $\mathfrak{g}$-modules, a linear map $f : V \to W$ is a homomorphism of $\mathfrak{g}$-modules if $f(g \cdot v) = g \cdot f(v)$ for every $g \in \mathfrak{g}$, $v \in V$. We write $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ for the set of all $\mathfrak{g}$-modules homomorphisms from $V$ to $W$.

Of course, if $z \in \mathbb{C}$ and $f$ is a $\mathfrak{g}$-module map, then $zf$ defined pointwise is also a $\mathfrak{g}$-module map. Therefore $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ is a $\mathbb{C}$-submodule of $\mathrm{Hom}_{\mathbb{C}}(V, W)$.

On the level of representations, an homomorphism of $\mathfrak{g}$-modules is exactly an intertwining map of representations.

An *equivalence of representations* is an isomorphism of $\mathfrak{g}$-modules, i.e. a $\mathfrak{g}$-module map which is an isomorphism of complex vector spaces.

Definition 1.2.9. The $\mathfrak{g}$-modules and $\mathfrak{g}$-modules homomorphisms form a category denoted by $\mathrm{Mod}(\mathfrak{g})$.

By the above remarks, the category $\mathrm{Mod}(\mathfrak{g})$ is additive. It is an easy exercise to show that it is in fact an abelian category.

Assume that $\mathfrak{g}$ is semisimple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$.

If $\lambda$ is a weight, then the subspace

$$V_\lambda = \{v \in V \ : \ \rho(h)v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$$

is not trivial and is called the *weight space* associated to $\lambda$. It is a basic and well-known result that every finite dimensional representation is the direct sum of all its weight spaces.

A source of finite dimensional representations of Lie algebras are Lie group representations. Suppose that $G$ is a complex Lie group with Lie algebra $\mathfrak{g}$. A *representation* of the Lie group $G$ is the data of a complex vector space $V$ together with a holomorphic morphism $\rho : G \to \mathrm{GL}(V)$. Theorems of I.D. Ado (see [**1, 2**]) and K. Iwasawa (see [**43**]) allows to prove that every connected Lie group has a finite dimensional representation. In a way similar to Lie algebras, a morphism between two representations of a same Lie group is a linear map which commutes with the action of $G$. In section 1.2, we explained how representations of Lie groups corresponds to representations of Lie algebras. In particular, proposition 1.2.4 shows that when $G$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, there is a correspondence between representations of the Lie group $G$ and representations of the Lie algebra $\mathfrak{g}$ induced by the functor $\mathscr{L}ie$ defined p.9

Similarly to the Lie algebra representations setting, if $G$ is a simply connected semisimple complex Lie group with $\mathfrak{g} = \mathscr{L}ie(G)$, we introduce the category $\mathrm{Mod}(G)$ as follows:

Definition 1.2.10. Denote by $\mathrm{Mod}(G)$ the category of $G$-modules equivalent to $\mathrm{Mod}(\mathfrak{g})$ via the $\mathscr{L}ie$ functor.

**1.2.2. Highest weight representations.** We will now concentrate on irreducible representations of semisimple complex Lie algebras. In subsection 1.3.1, these concepts will be generalized to parabolic subalgebras. Recall that a representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is *irreducible* if there is no proper invariant subspace of $V$. It is called *completely reducible* if $V$ is isomorphic to a direct sum of irreducible representations. Using the module terminology, a $\mathfrak{g}$-module $V$ is said *irreducible* if it has precisely two $\mathfrak{g}$-submodules (itself and $\{0\}$). Denote by $\mathcal{U}(\mathfrak{b})$ the universal enveloping algebra of a Lie algebra $\mathfrak{b}$.

The definition 1.1 gives a total ordering on $\mathfrak{h}^*$. It also induces the notion of highest weight of a representation as follows.

DEFINITION 1.2.11. Let $V$ be a representation of $\mathfrak{g}$. A weight $\lambda$ of $V$ is *highest* in $V$ (resp. *lowest* in $V$) if $\lambda \succ \mu$ (resp. $\lambda \prec \mu$) for every weight $\mu$ different from $\lambda$.

The following results about weights of representations are well-known.

PROPOSITION 1.2.12. *A finite dimensional irreducible representation $\rho$ : $\mathfrak{g} \to \mathfrak{gl}(V)$ of $\mathfrak{g}$ has a unique highest weight and the associated weight $\lambda$ is such that $(\lambda, \alpha_i^\vee) \in \mathbb{N}$ for every $i \in I_0$. Moreover, the highest weight $\lambda$ of the given representation has the following properties :*

1. *it depends only on the simple system;*
2. *the weight space $V_\lambda$ is 1-dimensional;*
3. *each element of $\mathfrak{g}_\alpha$ with $\alpha \in \Delta^+$ annihilates the elements of $V_\lambda$ and the members of $V_\lambda$ are the only vectors with this property.*
4. *every weight of the representation $\rho$ can be written*
$$\lambda - \sum_{i \in I_0} n_i \, \alpha_i$$
*where $n_i \in \mathbb{N}$ ($i \in I_0$).*
5. *if $\mu$ is a weight for $\rho$, then all weight spaces associated to the elements of the orbit of $\mu$ under the action of the Weyl group of $\mathfrak{g}$ have the same dimension: we have $\dim V_{w\mu} = V_\mu$ for all $w \in W_\mathfrak{g}$.*
6. *each weight $\mu$ of $\rho$ satisfies $|\mu| \leq |\lambda|$ and the equality stands only if $\mu$ belongs to the orbit $W_\mathfrak{g}\lambda$.*

PROOF. See **[87]**, lemmas 4.6.5 p.317 and 4.6.9 p.322. □

PROPOSITION 1.2.13. *If $\lambda$ is a weight of a Lie algebra representation then $(\lambda, \alpha^\vee)$ is an integer for all $\alpha \in \Delta$.*

PROOF. See **[87]**, lemma 4.6.9 p.322. □

Then it is natural to introduce the following definition:

DEFINITION 1.2.14. An element $\lambda$ of $\mathfrak{h}^*$ is *integral for* $\mathfrak{g}$ if $(\lambda, \alpha^\vee)$ is an integer for all $\alpha \in \Delta$. The lattice constituted by integral elements of $\mathfrak{g}$ is denoted by $\mathfrak{h}^*_\mathbb{Z}$.

In order to check the integrality of an element, it suffices to check that $(\lambda, \alpha^\vee)$ is an integer only for simple roots $\alpha$. More precisely, by decomposing roots of $\mathfrak{g}$ as linear combinations of simple roots with integral coefficients, we can prove that if $\lambda$ is an element of $\mathfrak{h}^*$ such that $(\lambda, \alpha_i^\vee) \in \mathbb{Z}$ for every $i \in I_0$, then $\lambda$ is integral for $\mathfrak{g}$.

Moreover, the set of integral elements is invariant under the action of the Weyl group as it is shown in the following result:

Proposition 1.2.15. *If $\lambda$ is an integral for $\mathfrak{g}$ and $w \in W_{\mathfrak{g}}$, then $w\lambda$ is integral for $\mathfrak{g}$.*

Proof. Let $\alpha \in \Delta$. Since a direct computation shows that $w^{-1}\alpha^\vee = (w^{-1}\alpha)^\vee$, we obtain that $(w\lambda, \alpha^\vee) = (\lambda, (w^{-1}\alpha)^\vee)$ and conclusion follows from proposition 1.1.12. □

Definition 1.2.16. An element $\lambda \in \Delta$ is said to be *dominant* (resp. *strictly dominant*) *for* $\mathfrak{g}$ if the coefficients $(\lambda, \alpha_i^\vee)$ $(i \in I_0)$ are non-negative real numbers (resp. positive real numbers). The subset of $\mathfrak{h}^*_{\mathbb{Z}}$ constituted by elements which are dominant for $\mathfrak{g}$ is denoted by $(\mathfrak{h}^*_{\mathbb{Z}})_{I_0}$ and is called the *positive Weyl chamber.*

The following result is the classical correspondence between dominant integral weights and finite dimensional irreducible representations. The interested reader can find an explicit proof in [**42**] or [**87**], theorem 4.7.1 p.324.

Proposition 1.2.17. *Up to equivalence, the finite dimensional irreducible representations of $\mathfrak{g}$ are in one-to-one correspondence with the integral and dominant weights for $\mathfrak{g}$.*

One direction of the correspondence just stated is given by the map which assigns to each finite dimensional irreducible representation its highest weight (whose existence and unicity is guaranteed by proposition 1.2.12). The other direction consists in the construction of a finite dimensional irreducible representation $V_{\mathfrak{g}}(\lambda)$ of $\mathfrak{g}$ from a given weight $\lambda$ which is integral and dominant for $\mathfrak{g}$. This may be achived using Verma modules as follows (or also using the Bott-Borel-Weil theorem 1.6.1).

Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I_0}$ and denote by $\mathbb{C}_\lambda$ the complex vector space $\mathbb{C}$ endowed with the action of $\mathfrak{h}$ defined by

$$h \cdot z = \lambda(h)z$$

for $h \in \mathfrak{h}$, $z \in \mathbb{C}$. Thus $\mathbb{C}_\lambda$ is a representation of $\mathfrak{h}$. Set

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$$

and denote by $\mathfrak{b}^+$ the Lie subalgebra of $\mathfrak{g}$ defined by $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$. This subalgebra is called a *Borel subalgebra* of $\mathfrak{g}$ and more precisions about such subalgebras (and some generalization as parabolic subalgebras) will be given in section 1.3. Letting $\mathfrak{n}^+$ act trivially on $\mathbb{C}_\lambda$, we obtain a representation $\mathbb{C}_\lambda$ of $\mathfrak{b}^+$ and hence a $\mathcal{U}(\mathfrak{b}^+)$-module. For the reader familiar with the notion of induced representation, the Verma module associated to $\lambda$ is the $\mathfrak{g}$-module induced by $\mathbb{C}_\lambda$. Explicitely, we have:

DEFINITION 1.2.18. The *Verma module associated to* $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I_0}$ is

$$M_{I_0}(\lambda) = \mathrm{ind}_{\mathcal{U}(\mathfrak{b})}^{\mathcal{U}(\mathfrak{g})}(\mathbb{C}_\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

Verma modules have the following elementary properties ([**21**] or [**55**]) :

(1) $M_{I_0}(\lambda)$ is a representation of $\mathfrak{g}$ of highest weight $\lambda$ and has a unique irreducible quotient denoted by $V_{I_0}(\lambda)$;
(2) each weight space of $M_{I_0}(\lambda)$ is finite dimensional.

Thus the remaining direction of the correspondence in proposition 1.2.17 assigns $V_{I_0}(\lambda)$ to $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_{I_0}$.

Note that the convention chosen here differs from the one chosen in [**55**], where the action on $M_{I_0}(\lambda)$ (and hence its the associated highest weight) is given by $\lambda - \delta$ where $\delta$ is the Weyl vector of $\mathfrak{g}$ (see p8). This latter convention is convenient when the affine Weyl action is not needed. In our context, the affine action of the Weyl group will be a key tool and this is the reason why we define the Verma module $M_{I_0}(\lambda)$ this way.

## 1.3. Parabolic subalgebras and parabolic subgroups

Now we focus our attention to a certain class of subalgebras of $\mathfrak{g}$. The simplest of them are called Borel subalgebras of $\mathfrak{g}$.

DEFINITION 1.3.1. A *Borel subalgebra* of $\mathfrak{g}$ is a maximal solvable subalgebra of $\mathfrak{g}$.

It is easy to see that

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$$

are nilpotent subalgebras of $\mathfrak{g}$. Then $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$ are two examples of Borel subalgebras of $\mathfrak{g}$. The subalgebra $\mathfrak{b}^+$ is called the *standard* Borel subalgebra.

In the case where $\mathfrak{g} = \mathrm{sl}(n, \mathbb{C})$, the subalgebra $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is the algebra of upper (resp. lower) triangular matrices.

Actually, one can show that all the Borel subalgebras of $\mathfrak{g}$ are conjugate to the standard Borel subalgebra of $\mathfrak{g}$:

PROPOSITION 1.3.2. *Any Borel subalgebra of* $\mathfrak{g}$ *is conjugate to the standard Borel subalgebra of* $\mathfrak{g}$ *defined by* $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ *where*

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

PROOF. See [**42**] pp.84–86. □

Let us introduce a more general class of subalgebras of $\mathfrak{g}$.

DEFINITION 1.3.3. A subalgebra of $\mathfrak{g}$ is a *parabolic subalgebra* of $\mathfrak{g}$ if it contains a Borel subalgebra.

There is a easy way to obtain many examples of parabolic subalgebras of $\mathfrak{g}$, generalizing the construction of Borel subalgebras previously given. Let $I$ be a subset of $I_0$, and define

$$\Delta_I = \Delta \cap \sum_{i \in I} \mathbb{Z}\alpha_i, \quad \Delta_I^\pm = \Delta_I \cap \Delta^\pm.$$

Moreover, define

$$\mathfrak{l}_I = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha, \quad \text{and } \mathfrak{u}_I^\pm = \bigoplus_{\alpha \in \Delta^\pm \backslash \Delta_I} \mathfrak{g}_\alpha.$$

The subalgebra $\mathfrak{l}_I$ of $\mathfrak{g}$ is reductive and $\mathfrak{u}_I^\pm$ are nilpotent. We can see that $\mathfrak{q}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{u}_I^\pm$ contains $\mathfrak{b}^\pm$ and hence are parabolic subalgebras of $\mathfrak{g}$. We say that $\mathfrak{q}_I^+$ is the *standard* parabolic subalgebra of $\mathfrak{g}$ associated to $I$. The decomposition $\mathfrak{q}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{u}_I^\pm$ is called a « *Levi decomposition* » of $\mathfrak{q}_I^\pm$ because of the following result :

PROPOSITION 1.3.4. *Let* $\mathfrak{q}_I$ *be a standard parabolic subalgebra of* $\mathfrak{g}$ *associated to a subset* $I$ *of* $I_0$ *and let* $\mathfrak{l}_I$ *and* $\mathfrak{u}_I^\pm$ *be defined as above. Then*

(1) $\mathfrak{l}_I$ *is a reductive subalgebra of* $\mathfrak{g}$*;*
(2) $\mathfrak{u}_I^\pm$ *is an ideal in* $\mathfrak{q}_I$*.*

PROOF. See [**44**] for a proof. □

The subalgebra $\mathfrak{l}_I$ in the decomposition $\mathfrak{q}_I^\pm = \mathfrak{l}_I \oplus \mathfrak{u}_I^\pm$ is called *Levi factor of* $\mathfrak{q}_I$. The proposition here above shows that Levi factors are always reductive, but they are usually not semisimple.

The extremal cases of parabolic subalgebras are $\mathfrak{q}_\emptyset^\pm = \mathfrak{b}^\pm$ and $\mathfrak{q}_{I_0} = \mathfrak{g}$.

It will be useful in the sequel (see corollary 1.3.7) to remark that since $\Delta_I^\pm = \Delta^\pm \cap \Delta_I$ by definition, we obtain

$$\Delta^\pm \backslash \Delta_I^\pm = \Delta^\pm \backslash \Delta_I. \tag{4}$$

We can adapt the previous definitions to parabolic subalgebras. Suppose that $I$ is a subset of $I_0$.

DEFINITION 1.3.5. The subgroup of the Weyl group $W_{\mathfrak{g}}$ generated by the identity and the compositions of reflections $\sigma_\alpha$ ($\alpha \in \Delta_I$) is denoted by $W_I$. We call $W_I$ the *Weyl group of* $\mathfrak{q}_I$.

Of course, the notation has been introduced in order to have $W_{I_0} = W_{\mathfrak{g}}$ (see definition 1.1.13). Exactly as in the global case, the Weyl chambers of $\mathfrak{q}_I$ are the connected components of $\mathfrak{h}^*_{\mathbb{R}} \backslash \bigcup_{\alpha\in\Delta_I} W_\alpha$ and the Weyl group of $\mathfrak{q}_I$ acts simply transitively on Weyl chambers of $\mathfrak{q}_I$. There is a unique element $w_I$ of $W_I$ such that $w_I \Delta_I^- = \Delta_I^+$, which is the longest in $W_I$ i.e. which satisfies $\ell(w) < \ell(w_I)$ for every $w \in W_I \backslash \{w_I\}$.

The following lemma will be useful in the proof of corollary 1.3.7. Its proof is classical and can be read in many books cited in the introduction of this chapter.

LEMMA 1.3.6. *The length of $w_I$ is the number of positive roots in $\Delta_I$ i.e. we have $\ell(w_I) = \#\Delta_I^+$.*

Let us define

$$\delta_I = \frac{1}{2} \sum_{\alpha\in\Delta_I^+} \alpha$$

so that $\delta_{I_0}$ is equal to $\delta$ that we have introduced p.8. Also let

$$\delta(\mathfrak{u}_I) = \frac{1}{2} \sum_{\alpha\in\Delta^+\backslash\Delta_I} \alpha. \tag{5}$$

Of course, we have

$$(\delta(\mathfrak{u}_I),\, \alpha) = 0 \tag{6}$$

for every $\alpha \in \Delta_I$ and we verify that $\delta(\mathfrak{u}_I)$ is invariant under the action of the Weyl group $W_I$, i.e. we have $w\delta(\mathfrak{u}_I) = \delta(\mathfrak{u}_I)$ for every $w \in W_I$.

Now we prove an easy result which will be very important for expressing some shifts of cohomology complexes in chapter 2.

COROLLARY 1.3.7. *If $I \subset I_0$, the complex dimension of the standard parabolic subalgebra of $\mathfrak{g}$ associated to $I$ is equal to*

$$\dim^{\mathbb{C}} \mathfrak{q}_I = \dim^{\mathbb{C}} \mathfrak{h} + \#\Delta^+ + \#\Delta_I - \ell(w_I).$$

PROOF. It follows directly from lemma 1.3.6. □

DEFINITION 1.3.8. An element $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is *affinely singular for* $\mathfrak{q}_I$ if it has a non-trivial stabilizer under the affine action of the Weyl group of $\mathfrak{q}_I$.

Once again, an element $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$ is affinely singular for $\mathfrak{q}_I$ if and only if there is $\alpha \in \Delta_I$ such that $(\lambda + \delta,\, \alpha) = 0$.

DEFINITION 1.3.9. An element $\lambda \in \mathfrak{h}^*$ is *dominant* (resp. *strictly dominant*, *integral*) *for* $\mathfrak{q}_I$ if $(\lambda, \alpha_i^\vee)$ is a non-negative real number (resp. positive real number, integer) for every $i \in I$.

There is also a relative version of Verma modules constructed on p.15. When $\mathfrak{q}_I$ is a standard parabolic algebra associated to a selection $I \subset I_0$ of simple roots, denote by $\mathfrak{q}_I = \mathfrak{l}_I \oplus u_I$ the Levi decomposition of $\mathfrak{q}_I$ where $\mathfrak{l}_I$ is a reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{u}_I$ is an ideal of $\mathfrak{q}_I$ (see proposition 1.3.4). Let $V_I(\lambda)$ be an irreducible finite dimensional representation of $\mathfrak{l}_I$ with highest weight $\lambda$. Consider $V_I(\lambda)$ as a $\mathfrak{q}_I$-module by letting the action of $\mathfrak{u}_I$ on $V_I(\lambda)$ to be trivial. The *generalized Verma module* associated to $\lambda$ is the $\mathfrak{g}$-module defined by

$$M_I(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q}_I)} V_I(\lambda).$$

Let us come back to the structure of parabolic subalgebras and their classification. One can show that all parabolic subalgebras are conjugated to a standard parabolic subalgebra. More precisely, we have the following result which gives a way to construct all parabolic subalgebras:

PROPOSITION 1.3.10. *The parabolic subalgebras $\mathfrak{q}$ containing the standard Borel subalgebra $\mathfrak{b}$ are parametrized by the choice of a subset of the set of simple roots. In other words, if $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$, there is a bijection between the set of parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$ and the set of subsets of $\mathcal{S}$.*

PROOF. If we are given a parabolic subalgebra $\mathfrak{q}$ which contains $\mathfrak{b}$, we can write

$$\mathfrak{q} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Gamma(\mathfrak{q})} \mathfrak{g}_\alpha$$

where $\Gamma(\mathfrak{q})$ is a subset of $\Delta$ containing $\Delta^+$. Now define $I_\mathfrak{q}$ to be the subset of $I_0$ whose elements correspond to simple roots of $\mathfrak{g}$ which belong to the linear span of $\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$. Then $\mathfrak{q} \mapsto I_\mathfrak{q}$ is a map from the set of parabolic subalgebras containing $\mathfrak{b}$ to the set of subsets of $I_0$. In the reverse direction, if $I \subset I_0$ is given, define $\mathfrak{q}_I^+$ to be the standard parabolic subalgebra associated to $I$. To complete the proof, we have to show that these two maps are inverse each to one another.

First, we are going to show that $I_{q_I^+} = I$ for every $I \subset I_0$. We have $\Gamma(\mathfrak{q}_I^+) = \Delta^+ \cup \Delta_I$ and since $\Delta_I$ is closed under negatives, we get

$$\Gamma(\mathfrak{q}_I^+) \cap -\Gamma(\mathfrak{q}_I^+) = (\Delta^+ \cup \Delta_I) \cap (\Delta^- \cup \Delta_I) = \Delta_I.$$

Thus, since the $\alpha_i$ $(i \in I_0)$ are linearly independant, we obtain that the simple roots in the span of $\Gamma(\mathfrak{q}_I^+) \cap -\Gamma(\mathfrak{q}_I^+)$ are precisely the simple roots corresponding to $\alpha_i$ with $i \in I$.

Let us now prove that $\mathfrak{q}_{I_\mathfrak{q}}^+ = \mathfrak{q}$ when $\mathfrak{q}$ is a parabolic subalgebra which contains $\mathfrak{b}$. We are going to show that $\Gamma(\mathfrak{q}_{I_\mathfrak{q}}^+) = \Gamma(\mathfrak{q})$. To begin, we already know that $\Gamma(\mathfrak{q}_{I_\mathfrak{q}}^+) = \Delta^+ \cup \Delta_{I_\mathfrak{q}}$ and since $\Delta^+$ is a subset of $\Gamma(\mathfrak{q})$, we have to prove that $\Delta_{I_\mathfrak{q}} \subset \Gamma(\mathfrak{q})$ and $\Gamma(\mathfrak{q}) \subset \Delta^+ \cup \Delta_{I_\mathfrak{q}}$. Since $\Gamma(\mathfrak{q})$ is equal to $\Delta^+ \cup (\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q}))$, the latter inclusion will follow if we establish that $\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$ is a subset of $\Delta^+ \cup \Delta_{I_\mathfrak{q}}$.

To see that $\Delta_{I_\mathfrak{q}} \subset \Gamma(\mathfrak{q})$, take $\alpha \in \Delta_{I_\mathfrak{q}}$. If $\alpha \in \Delta^+$, it is obvious that $\alpha \in \Gamma(\mathfrak{q})$. Now suppose furthermore that $\alpha \in \Delta^-$ and write

$$\alpha = \sum_{i \in I_\mathfrak{q}} n_i \, \alpha_i$$

where each $n_i$ $(i \in I_\mathfrak{q})$ is non-positive. Since

$$0 < |\alpha|^2 = \sum_{i \in I_\mathfrak{q}} n_i \, (\alpha_i, \, \alpha),$$

there is an element $\nu(1)$ of $I_\mathfrak{q}$ such that $(\alpha_{\nu(1)}, \, \alpha) < 0$. The $\alpha_{\nu(1)}$–string containing $\alpha$ is given by

$$\{\alpha + n\alpha_{\nu(1)} \; : \; p \leq n \leq q\}$$

where $p \leq q$ are two integers such that

$$p + q = \frac{-2}{|\alpha_{\nu(1)}|^2} \, (\alpha, \, \alpha_{\nu(1)}).$$

Since $(\alpha_{\nu(1)}, \, \alpha) < 0$, we obtain that $q \geq 1$ and then $\alpha + \alpha_{\nu(1)} \in \Delta^-$. Then $e_\alpha$ is a multiple of $\mathrm{ad}(e_{-\alpha_{\nu(1)}}) \, e_{\alpha + \alpha_{\nu(1)}}$. Iterating this reasoning, we obtain an integer $p$ and some $\nu(j) \in I_\mathfrak{q}$ $(1 \leq j \leq p+1)$ such that $e_\alpha$ is a multiple of

$$\mathrm{ad}(e_{-\alpha_{\nu(1)}}) \circ \ldots \circ \mathrm{ad}(e_{-\alpha_{\nu(p)}}) \, e_{-\alpha_{\nu(p+1)}}.$$

Since $e_{-\alpha_{\nu(j)}}$ belongs to $\mathfrak{q}$ for every $1 \leq j \leq p+1$, we obtain that $e_\alpha \in \mathfrak{q}$ and this shows that $\alpha \in \Gamma(\mathfrak{q})$.

Now we have to prove that $\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$ is a subset of $\Delta^+ \cup \Delta_{I_\mathfrak{q}}$. Since $\Delta^+ \subset \Delta_{I_\mathfrak{q}}$, we only have to show that $\Delta^- \cap \Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$ is included in $\Delta_{I_\mathfrak{q}}$. Take some $\beta \in \Delta^- \cap \Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$ and write

$$-\alpha = \sum_{i \in I_0} -n_i \, \alpha_i$$

where $n_i$ $(i \in I_0)$ are non-negative integers. The assertion is that each $\alpha_i$ for which $n_i > 0$ is in $\Delta_{I_\mathfrak{q}}$. Let us prove this by induction on $\sum_{i \in I_0} n_i$. The case of the sum being equal to one is trivial by definition of $I_\mathfrak{q}$. Now suppose that $\sum_{i \in I_0} n_i > 1$. This means that the decomposition

$$\alpha = \sum_{i \in I_0} n_i \, \alpha_i$$

contains at least two factors where all coefficients are non-negative. Hence we can write $\alpha = \beta + \gamma$ where $\beta, \gamma \in \Delta^+$. Since $-\alpha \in \Gamma(\mathfrak{q})$ and $\beta, \gamma \in \Delta^+$, we get that $e_\alpha$, $e_\beta$ and $e_\gamma$ are in $\mathfrak{q}$. Then since $[e_{-\alpha}, e_\beta] \in \mathfrak{q}$ is a multiple of $e_{-\gamma}$, we obtain that $e_{-\gamma} \in \mathfrak{q}$. Similarly, we obtain that $e_{-\beta} \in \mathfrak{q}$. So $-\beta$ and $-\gamma$ are elements of $\Gamma(\mathfrak{q}) \cap -\Gamma(\mathfrak{q})$ and induction gives that the constituent simple roots of $-\beta$ and $-\gamma$ are indexed by integers which belongs to $I_\mathfrak{q}$ and thus the same thing is true for $-\alpha = -\beta - \gamma$. This concludes the proof. $\square$

Hence all parabolic subalgebras of $\mathfrak{g}$ can be written (up to conjugation) like $\mathfrak{q}_I$ with $I$ being a subset of $I_0$.

**1.3.1. Parabolic subalgebras and their highest weight representations.** Now we generalize and relativize all the results of section 1.2.2 to representations of parabolic subalgebras. Let $I$ be a subset of $I_0$ and consider the associated standard parabolic subalgebra $\mathfrak{q}_I$ of $\mathfrak{g}$ defined in p16.

Parabolic subalgebras provide a framework for generalizing the theorem of highest weight so that the Cartan subalgebra is replaced by the Levi factor of the parabolic subalgebra. More precisely, let $\mathfrak{q}$ be a parabolic subalgebra with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ where $\mathfrak{l}$ is reductive and $\mathfrak{u}$ nilpotent. By Engel's theorem, the nilpotent part $\mathfrak{u}$ acts trivially on any irreducible representation of $\mathfrak{q}$ since $\mathfrak{u}$ acts by nilpotent endomorphisms. Thus an irreducible representation of $\mathfrak{q}$ corresponds to an irreducible representation of $\mathfrak{l}$. One can show that in any representation of $\mathfrak{l}$ for which the Cartan subalgebra acts completely reducibly, then $\mathfrak{l}$ acts also completely reducibly and this happens when the action of a representation of $\mathfrak{g}$ is restricted to $\mathfrak{l}$ (see **[55]** p.331). Moreover, each irreducible constituent of such completely reducible representation consists of a scalar action by the center of $\mathfrak{l}$ (multiplication by a character) and an irreducible representation of the semisimple part of $\mathfrak{l}$. So the theorem of the highest weight is applicable for the semisimple part of $\mathfrak{l}$ and may be reinterpreted as valid for $\mathfrak{l}$.

So just like in the global case, the irreducible finite dimensional representations of $\mathfrak{q}_I$ are in correspondence with the elements of $\mathfrak{h}^*$ which are dominant and integral for $\mathfrak{q}_I$ (see **[7]** p.23). If $\lambda$ is an element of $\mathfrak{h}^*$ dominant and integral for $\mathfrak{q}_I$, we denote by $V_I(\lambda)$ the associated irreductible finite dimensional representation of $\mathfrak{l}_I$ and we let $\mathfrak{u}_I$ acting on it trivially so that $V_I(\lambda)$ becomes an irreducible finite dimensional representation of $\mathfrak{q}_I$. In the other direction, the correspondence assigns to an irreducible finite dimensional representation of $\mathfrak{q}_I$ its highest weight.

In fact, we will be interested in the representations of Lie groups rather than representations of Lie algebras. The classification of representations of Lie groups corresponds to the one for Lie algebras for semisimple simply connected complex Lie groups (see proposition 1.2.4). However, if the parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ corresponds to the subgroup $Q$ of $G$, the necessary and sufficient condition on highest weight $\lambda$ of a given finite dimensional irreducible representation of $\mathfrak{q}$ in order to exponentiate to one for $Q$ is that $\lambda$ is dominant for $\mathfrak{q}$ and integral *for* $\mathfrak{g}$ and not just for $\mathfrak{q}$. This is the reason why we introduce the following notation:

DEFINITION 1.3.11. The subset of $\mathfrak{h}_{\mathbb{Z}}^*$ constituted by elements which are dominant for $\mathfrak{q}_I$ is denoted by $(\mathfrak{h}_{\mathbb{Z}}^*)_I$.

For $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$, we will denote by $V_{Q_I}(\lambda)$ (or simply $V_I(\lambda)$ if there is no possible confusion) the irreducible finite dimensional representation of $Q_I$ with highest weight $\lambda$.

Since there is a close link between Lie algebras and Lie groups through the $\mathscr{L}ie$ functor, there is a analog of the parabolic subalgebras for Lie subgroups. We will now introduce what are Borel and parabolic subgroups, their classification and we will establish the natural link between parabolic subgroups and parabolic subalgebras.

### 1.3.2. Borel and parabolic subgroups.

DEFINITION 1.3.12. A *Borel subgroup* of $G$ is a maximal connected solvable closed subgroup of $G$.

Denote by $G^\circ$ the connected component of $G$ at identity. Since Borel subgroups of $G$ and $G^\circ$ coincide, we shall always assume in what follows that $G$ is connected.

PROPOSITION 1.3.13. *A closed subgroup $B$ of $G$ is a Borel subgroup if and only if $\mathscr{L}ie(B)$ is a Borel subalgebra of $\mathscr{L}ie(G)$.*

PROOF. If $B$ is a Borel subgroup of $G$, its associated Lie algebra $\mathscr{L}ie(B)$ is a maximal solvable subalgebra of $\mathscr{L}ie(G)$ since the equivalence $\mathscr{L}ie$ preserves these properties. Conversely, if we suppose that $\mathscr{L}ie(B)$ is maximal solvable subalgebra of $\mathscr{L}ie(G)$, the connectedness of $G$ gives that the connected subgroup of $G$ associated to $\mathscr{L}ie(B)$ is precisely $B$ and is also maximal solvable in $G$. □

In view of this correspondence, the subgroup of $G$ whose Lie algebra is the standard Borel subalgebra of $\mathscr{L}ie(G)$ will be called the *standard Borel subgroup* of $G$.

It is obvious that a connected solvable closed subgroup of $G$ of largest possible dimension is a Borel subgroup. This a a consequence of the following result (see **[41]** p.134) :

THEOREM 1.3.14. *If $B$ is a Borel subgroup of an algebraic group $G$, then the quotient $G/B$ is a projective variety. Moreover, all other Borel subgroups of $G$ are conjugate to $B$.*

Before going into its proof, we recall the following algebraic fixed point theorem

THEOREM 1.3.15 (A. BOREL [**14**]). *If $H$ is a connected, solvable algebraic group acting regularly on a non-empty, complete algebraic variety $V$ over an algebraically closed field $k$, then $H$ has a fixed point in $V$.*

PROOF OF THEOREM 1.3.14. Let $M$ be a Borel subgroup of $G$ of maximal dimension. Let $\rho : G \to \mathrm{GL}(V)$ be a finite dimensional representation of $G$ for which there is a one-dimensional subspace $L$ of $V$ such that $M = \mathrm{stab}_G(L)$. Since $M$ is connected, solvable and acts on $V/L$, we apply the Lie-Kolchin theorem to get a flag $\mathcal{F} = (L \subset \mathcal{F}_2 \subset \ldots \subset V)$ which is stabilized under the action of $M$. Since $M$ is include in $\mathrm{stab}_G(\mathcal{F})$ which is include in $\mathrm{stab}_G(L) = M$, we deduce that $M = \mathrm{stab}_G(\mathcal{F})$. Hence the orbit map induces a bijective morphism $G/M \to G\mathcal{F}$. On the other hand, the stabilizer of any flag is solvable since it is upper triangular in some basis, and therefore has dimension not bigger than $M$. It follows that the orbit $G\mathcal{F}$ has the smallest possible dimension, hence is closed. This forces the orbit to be complete and the surjectivity of the morphism $G/M \to G\mathcal{F}$ finally gives that $G/M$ is complete. Since $G/M$ is quasi-projective, we get that $G/M$ is projective.

Now if $B$ is a given Borel subgroup of $G$, it acts by left multiplication on the complete variety $G/M$ so by the Borel's fixed point theorem 1.3.15, there is a fixed point $gM$ with $g \in G$. Since $BgM = gM$, we obtain that $g^{-1}Bg \subset M$ and since each of these groups are Borel subgroups of $G$, maximality gives that $g^{-1}Bg = M$ and we obtain that all Borel subgroups are conjugate to $M$. Moreover, since $G/M$ is isomorphic to $G/B$, we get that $G/B$ is complete and since it is by construction a quasi-projective variety, it is also projective. □

The second point of the preceeding theorem could also be proved directly using proposition 1.3.2.

It is interesting to remark that if $B$ is a Borel subgroup of an algebraic group $G$, then the quotient $G/B$ is in fact the largest projective variety of the type $G/H$ when $H$ is a closed subgroup of $G$. Indeed, if $H$ is a closed subgroup of $G$ such that $G/H$ is projective, then theorem 1.3.15 gives that $B$ fixes a point and therefore has a conjugate in $H$, forcing $\dim^{\mathbb{C}} G/H \leq \dim^{\mathbb{C}} G/B$. And if $H$ is a closed subgroup of $G$ which contains $B$, the projection $G/B \to G/H$ is a surjective morphism from a complete variety and this implies that $G/H$ is complete too. But since all homogeneous spaces are quasi-projective by construction, we obtain that $G/H$ is projective.

This observation lead us to define a slightly more general class of closed subgroups of $G$:

DEFINITION 1.3.16. A *parabolic subgroup* of $G$ is a closed subgroup $P$ of $G$ such that $G/P$ is a projective (equivalently, complete) variety.

We have the following result which gives a kind of minimality criterion for parabolicity of subgroups of $G$.

PROPOSITION 1.3.17. *Let $Q$ be a closed subgroup of $G$. Then $Q$ is parabolic if and only if it contains a Borel subgroup of $G$.*

PROOF. Above, we already obtained that if $Q$ contains a Borel subgroup, then $Q$ is parabolic in $G$. Now suppose that $Q$ is parabolic in $G$ and let $B$ be a Borel subgroup of $G$. It acts on $G/Q$ with a fixed point by theorem 1.3.15 so there is a $g \in G$ such that $BgQ = gQ$. This implies that $g^{-1}BgQ = Q$ and then $Q$ contains the Borel subgroup $g^{-1}Bg$. □

Of course, there is a close link between the notion of parabolic subalgebra and the one of parabolic subgroups. It is natural to hope that parabolic subgroups of $G$ correspond to parabolic subalgebras of $\mathscr{L}ie(G)$. This is true when the $\mathscr{L}ie$ functor is an equivalence of categories.

PROPOSITION 1.3.18. *If $G$ is simply connected, the parabolic subgroups of $G$ correspond to parabolic subalgebras of $\mathscr{L}ie(G)$ via the equivalence of categories given by the $\mathscr{L}ie$ functor.*

PROOF. Let $Q$ a parabolic subgroup of $G$. The proposition 1.3.17 gives that there is a Borel subgroup $B$ of $G$ contained in $Q$. Since $B \subset Q \subset G$ are inclusions of Lie groups, we get that $\mathscr{L}ie(B) \subset \mathscr{L}ie(Q) \subset \mathscr{L}ie(G)$. Proposition 1.3.13 gives that $\mathscr{L}ie(B)$ is a Borel subalgebra of $\mathscr{L}ie(G)$ and it follows that $\mathscr{L}ie(Q)$ is a parabolic subalgebra of $\mathscr{L}ie(G)$.

Conversely, if $\mathfrak{q}$ is a parabolic subalgebra of $\mathscr{L}ie(G)$, let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ which is included in $\mathfrak{q}$, and take closed subgroups $B$ and $Q$ of $G$ such that $B \subset Q$, $\mathscr{L}ie(B) = B$ and $\mathscr{L}ie(Q) = Q$ to conclude the proof. □

Since parabolic subalgebras of $\mathscr{L}ie(G)$ are conjugated to a standard parabolic one, the correspondence between parabolic subalgebras and parabolic subgroups gives that all parabolic subgroups are conjugated to subgroups of the type $Q_I$ such that $\mathscr{L}ie(Q_I) = \mathfrak{q}_I$ with $I \subset I_0$. Such particular subgroups of $G$ whose Lie algebra is a standard parabolic subalgebra of $\mathscr{L}ie(G)$ are called *standard parabolic subgroups* of $G$.

PROPOSITION 1.3.19. *If $P$ is a parabolic subgroup of $G$, then $G/P$ is a compact complex manifold. Conversely, if $H$ is a closed subgroup of $G$ such that $G/H$ is a compact complex manifold, then the normalizer in $G$ of the connected component of $H$ containing the identity is a parabolic subgroup of $G$.*

Proof. Denote by $\mathfrak{g}$ the Lie algebra associated to $G$, i.e. $\mathfrak{g} = \mathscr{L}ie(G)$.

First we are going to prove that if $P$ is a parabolic subgroup of $G$, then $G/P$ is compact. We know that the Lie algebra $\mathfrak{g}$ associated to $G$ contains a compact real form $\mathfrak{k}$, that is a compact real Lie algebra such that its complexification is precisely $\mathfrak{g}$. Explicitely, a compact real form of $\mathfrak{g}$ can be defined by

$$\mathfrak{k} = i\mathfrak{h}_{\mathbb{R}} \oplus \bigoplus_{\alpha\in\Delta} \mathbb{R}(X_\alpha - X_{-\alpha}) \oplus \bigoplus_{\alpha\in\Delta} i\mathbb{R}(X_\alpha + X_{-\alpha})$$

where $X_\alpha$ are some elements of $\mathfrak{g}_\alpha \backslash \{0\}$ (see **[8]** p.82 or **[55]** chapter 5 for more details). Now consider $\mathfrak{g}$ as a real Lie algebra and set

$$\mathfrak{n}^{\pm} = \bigoplus_{\alpha\in\Delta^{\pm}} \mathfrak{g}_\alpha.$$

One can see that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ = \mathfrak{k} \oplus \mathfrak{h}_{\mathbb{R}} \oplus \mathfrak{n}^+$. Let $KAN$ be an Iwasawa decomposition of $G$ where $K$, $A$ and $N$ are connected real Lie subgroups of $G$ with Lie algebras $\mathfrak{k}$, $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{n}^+$ respectively. The Iwasawa decomposition for semisimple complex Lie groups says that the multiplication map $K\times A\times N \to G$ is a diffeomorphism onto (see **[19]** p.203, **[55]** theorem 6.46 or **[38]** p.239 for instance). Moreover, one can show that the connected real Lie subgroup $K$ of $G$ whose associated Lie algebra is $\mathfrak{k}$ is a maximal compact subgroup of $G$. Then, since $AN \subset B \subset P$, we have $G = KB$ and $K$ acts transitively on $G/P$ if $P$ is a subset of $B$.Then $G/P$ is diffeomorphic to $K/(K \cap P)$ and the compacity of $G/P$ follows.

The proof of the second part of the statement has been given by J. Tits **[86]** theorem 4.1. Here we shall only recall the sketch of the proof. Denote by $N$ the normalizer in $G$ of the connected component of $H$ containing the identity. Let $k$ be the complex dimension of $H$ and let $h$ be the point in the Grassmann manifold $\mathbb{G}_k(\mathfrak{g})$ which corresponds to the Lie subalgebra $\mathscr{L}ie(H)$ of $\mathfrak{g}$. The adjoint representation

$$\mathrm{Ad}_G \ : \ G \to \mathrm{GL}(\mathfrak{g})$$

composed with the natural action

$$\mathrm{GL}(\mathfrak{g}) \times \mathbb{G}_k(\mathfrak{g}) \to \mathbb{G}_k(\mathfrak{g})$$

defines an action of $G$ on $\mathbb{G}_k(\mathfrak{g})$ and the isotropy subgroup $G_h$ coincides with $N$ and since $H$ is a subset of $N$, the orbit $Z = G(h)$ is a compact complex manifold and hence is an algebraic subvariety of $\mathbb{G}_k(\mathfrak{g})$ by the Chow's theorem. Take $B$ a Borel subgroup of $G$ and let $A$ be the Zariski closure of $\mathrm{Ad}_G(B)$ in $\mathrm{GL}(\mathfrak{g})$. Since $\mathrm{Ad}_G(B)$ is a connected solvable Lie subgroup of $\mathrm{GL}(\mathfrak{g})$, it follows that $A$ is also solvable and connected. This implies that the subgroup

$$\{g \in \mathrm{GL}(\mathfrak{g}) \ : \ gZ = Z\} \subset \mathrm{GL}(\mathfrak{g})$$

is algebraic. Then the Borel's fixed point theorem 1.3.15 gives that $A$ has a fixed point on $Z$ and then $B$ also. Since $Z$ is $G$-homogeneous, there exists $g \in G$ such that $gBg^{-1} \subset G_h = N$ which shows that $N$ is a parabolic subgroup of $G$. □

## 1.4. Kähler manifolds

Among differential and algebraic geometers, Erich KÄHLER is widely known for having introduced the Kähler metrics in his article [**59**]. This paper was the starting point for Kähler geometry with its notions of Kähler manifolds and Kähler groups which are today fundamental ideas in string theory and the study of space-time. For instance, it is well known by theoretical physicists that the definition of a sypersymmetric non-linear sigma model requires the corresponding field space to be a Kählerian manifold (see [**90**]). In fact, it has been shown that in dimension four and in terms of superfields, the Lagrangian of the model is simply the Kähler potential. Thus homogeneous Kählerian manifolds arise quite naturally in modern theoretical physics. The reader interested in some uses of Kähler manifolds in superstrings theory will surely have fun reading [**52**].

We will denote by $\partial$ (resp. $\overline{\partial}$) the differentiation operator from $(p,\,0)$-forms to $(p+1,\,0)$-forms (resp. from $(0,\,p)$-forms to $(0,\,p+1)$-forms). In a local coordinates system $(z_1,\,\ldots,\,z_n)$ on $Z$, we have explicitely

$$\partial = \sum_{\alpha=1}^{n} \frac{\partial}{\partial z_\alpha} dz_\alpha \quad \text{and} \quad \overline{\partial} = \sum_{\alpha=1}^{n} \frac{\partial}{\partial \overline{z_\alpha}} d\overline{z}_\alpha.$$

The original motivation of E. KÄHLER was the following (technical justifications are detailed below). Given any hermitian metric $h$ on a complex manifold, we can express the associated fundamental two-form $\Omega$ in local holomorphic coordinates $(z_1,\,\ldots,\,z_n)$ by

$$\Omega = i \sum_{1\leq \alpha,\,\beta\leq n} h_{\alpha,\beta}\, dz_\alpha \wedge d\overline{z}_\beta$$

where

$$h_{\alpha,\beta} = h\left(\frac{\partial}{\partial z_\alpha},\, \frac{\partial}{\partial \overline{z}_\alpha}\right),\ 1 \leq \alpha,\,\beta \leq n.$$

He noticed that the exactness of the two-form $\Omega$ (i.e. $d\Omega = 0$) is equivalent to the local existence of some function $f$ which satisfies

$$h_{\alpha,\beta} = \frac{\partial^2 f}{\partial z_\alpha\, \partial \overline{z}_\beta}$$

for every $1 \leq \alpha,\,\beta \leq n$. In other words, the whole metric tensor is defined by a unique function $f$, called the *Kähler potential.* This remarkable property of the metric allows to give a simple explicit expression for the Christoffel symbols and the Ricci tensor.

DEFINITION 1.4.1. A *Hermitian metric* on a complex vector bundle $V$ over a smooth manifold $M$ is a smooth form which is a positive definite Hermitian form on each fiber. A *Hermitian manifold* is a complex manifold with a Hermitian metric on its holomorphic tangent space.

If $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system, a Hermitian metric on $Z$ is a form of the type

$$h = \sum_{\alpha,\, \beta=1}^{n} h_{\alpha,\beta}\, dz_\alpha \otimes d\overline{z_\beta}$$

where $H = (h_{\alpha,\beta})_{\alpha,\beta=1,\ldots,n}$ is a $n \times n$ matrix of smooth functions which is Hermitian symmetric and positive definite.

The real part of an Hermitian metric $h$ on a complex manifold defines a Riemannian metric on the underlying real manifold $Z_{\mathbb{R}}$, ie. a symmetric and bilinear form on the complexified tangent bundle $(TZ_{\mathbb{R}})^{\mathbb{C}}$. The underlying real manifold of an Hermitian manifold is thus a Riemannian manifold. Moreover, if we start with a Riemannian metric $g$ on $Z_{\mathbb{R}}$, then it is coming from a Hermitian metric if and only if $g(JX,\, JY) = g(X, Y)$ for any real vector fields $x$ and $Y$, where $J$ denotes the almost complex structure of $Z$ (given by the multiplication by $i$ on $TZ$).

Under a local holomorphic coordinate system on $Z$ given by

$$z_\alpha = x_\alpha + i\, x_{n+\alpha}$$

for every $\alpha = 1, \ldots, n$, then $(x_1, \ldots, x_{2n})$ is a local differentiable coordinate system in $Z_{\mathbb{R}}$. Let us decompose the Riemannian metric $g$ on $Z_{\mathbb{R}}$ associated to $h$ by

$$g = \sum_{\alpha,\, \beta=1}^{2n} g_{\alpha,\beta}\, dx_\alpha \otimes dx_\beta.$$

Then if $H = (h_{\alpha,\beta})_{\alpha,\beta=1,\ldots,n} = A + iB$ is the matrix associated to $h$ with $A$, $B \in \mathbb{R}^n_n$, the matrix associated to $g$ is given by

$$G = (g_{\alpha,\beta})_{\alpha,\beta=1,\ldots,2n} = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

In particular, we obtain that $\det G = |\det H|^2$.

DEFINITION 1.4.2. The *fundamental form* associated to the Hermitian metric $h$ on $Z$ is locally defined by $\Omega_h = -\Im h$ and locally, we have

$$\Omega_h = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}\, dz_\alpha \wedge d\overline{z_\beta}$$

where $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system on $Z$.

So $\Omega_h$ is a global (1, 1)-form on $Z$. Moreover, the fundamental form of a Hermitian metric is also real (i.e. $\overline{\Omega_h} = \Omega_h$) and positive since $h$ is positive. Conversely, given a global (1, 1)-form $\omega$ on $Z$, we can define a Hermitian metric $h$ by setting

$$h(X, Y) = -2i\,\omega(X, \overline{Y})$$

where $\overline{Y}$ denotes the conjugate of $Y$ in $(TZ_{\mathbb{R}})^{\mathbb{C}}$. If $\omega$ is positive, then $h$ is positive too and having a Hermitian metric on $Z$ is the same as having a global positive (1, 1)-form.

There is an important class of Hermitian manifolds, which are the Kähler manifolds.

Proposition 1.4.3. *If $Z$ is a complex manifold with Hermitian metric $h$, then the following assertions are equivalent:*

(1) *the fundamental form associated to $h$ is closed, i.e. $d\Omega_h = 0$;*
(2) *if $(z_1, \ldots, z_n)$ is a local holomorphic coordinate system on $Z$, then*

$$\frac{\partial h_{\alpha,\beta}}{\partial z_\gamma} = \frac{\partial h_{\gamma,\beta}}{\partial z_\alpha}$$

*for every $1 \leq \alpha, \beta, \gamma \leq n$.*

Proof. This follows from some computations on local coordinates systems. Details can be obtained for instance in [**89**]. □

Proposition 1.4.4. *If $Z$ is a complex manifold with hermitian metric $h$, then the Hermitian volume element $dV$ on $Z$ coincides with*

$$dV = \frac{1}{n!} \bigwedge^n \Omega_h$$

*and is locally given by*

$$dV = \det h \; dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n$$

*where $(z_1, \ldots, z_n)$ are local holomorphic coordinates on $Z$ and $z_\alpha = x_\alpha + iy_\alpha$ ($1 \leq \alpha \leq n$).*

Proof. Since $h$ is symmetric, the decomposition $z_\alpha = x_\alpha + iy_\alpha$ $(1 \leq \alpha \leq n)$ and a simple calculation leads to

$$\Omega_h = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}\, dz_\alpha \wedge d\overline{z_\beta} = \sum_{\alpha,\beta=1}^{n} h_{\alpha,\beta}\, dx_\alpha \wedge dy_\beta$$

and

$$\bigwedge^n \Omega_h = \sum_{\alpha_1,\beta_1=1}^{n} \cdots \sum_{\alpha_n,\beta_n=1}^{n} h_{\alpha_1,\beta_1} \ldots h_{\alpha_n,\beta_n}\, dx_{\alpha_1} \wedge dy_{\beta_1} \wedge \ldots \wedge dx_{\alpha_n} \wedge dy_{\beta_n}.$$

By antisymmetry of exterior product, we obtain

$$\bigwedge^n \Omega_h = \sum_{\alpha,\beta} h_{\alpha(1),\beta(1)} \dots h_{\alpha(n),\beta(n)}\, dx_{\alpha(1)} \wedge dy_{\beta(1)} \wedge \dots \wedge dx_{\alpha(n)} \wedge dy_{\beta(n)}$$

where the sum is over all permutations $\alpha$, $\beta$ of the set $\{1, \dots, n\}$. Some permutations of the differentials in the exterior product gives that

$$\begin{aligned} \bigwedge^n \Omega_h &= \left( \sum_{\alpha,\beta} (-1)^{\operatorname{sgn}\alpha + \operatorname{sgn}\beta} h_{\alpha(1),\beta(1)} \dots h_{\alpha(n),\beta(n)} \right) d\omega \\ &= n! \left( \sum_{\substack{\beta \text{ perm. of} \\ \{1,\dots,n\}}} (-1)^{\operatorname{sgn}\beta} h_{1,\beta(1)} \dots h_{n,\beta(n)} \right) d\omega \\ &= n! \det h\ d\omega \end{aligned}$$

where $d\omega = dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$. Consequently, the form

$$dV = \frac{1}{n!} \bigwedge^n \Omega_h = \det h\ dx_1 \wedge dy_1 \wedge \dots \wedge dx_n \wedge dy_n$$

is positive and coincides with the Hermitian volume element of $Z$. □

DEFINITION 1.4.5. A Hermitian metric $h$ on $Z$ is called a *Kählerian metric* if one of the equivalent assertions of proposition 1.4.3 (and hence all of them) are satisfied. A complex manifold is called a *Kähler manifold* if it is endowed with a Kählerian metric; the fundamental form associated to its Kählerian metric is the *Kähler form* of $Z$.

If $h$ is a Kählerian metric on a complex manifold, the associated fundamental form is denoted by $\Omega_h$.

EXAMPLE 1.4.6. Here are some basic examples of Kähler manifolds.

(1) The Euclidian metric in $\mathbb{C}^n$ defined by

$$ds^2 = \sum_{\alpha=1}^n dz_\alpha \otimes d\bar{z}_\alpha$$

is Hermitian and the Kähler form associated has the expression

$$\Omega = \frac{i}{2} \sum_{\alpha=1}^n dz_\alpha \wedge d\bar{z}_\alpha.$$

Thus $d\Omega = 0$ and $\mathbb{C}^n$ is a Kähler manifold.

(2) If $Z$ is a Riemann surface and $\Omega$ denotes the Kähler form associated to a Hermitian structure of $Z$, then the degree of $d\Omega$ is 3 which is

strictly greater that $2 = \dim^{\mathbb{R}} Z$. Thus $d\Omega = 0$ and $Z$ is a Kähler manifold.

(3) Products of Kähler manifolds, endowed with the product complex structure and the product metric, are Kähler manifolds.

(4) A complex submanifold $X$ of a Kähler manifold $Y$ is Kählerian with respect to the induced metric since the Kähler form on $X$ is the restriction of the Kähler form on $Y$.

(5) A complex torus is a quotient of $\mathbb{C}^n$ by a lattice $\Gamma$ of even rank. Such torus are compact complex manifolds and every positive definite Hermitian form $\omega = \sum_{\alpha,\beta} h_{\alpha,\beta}\, dz_\alpha \wedge d\overline{z}_\beta$ with constant coefficients $(h_{\alpha,\beta})$ defines a Kähler metric on it.

More generally, the last example given above can be generalized this way:

PROPOSITION 1.4.7. *Let $X$ and $Y$ be complex manifolds. If $f : X \to Y$ is a holomorphic immersion and if $Y$ is a Kähler manifold, then $X$ is a Kähler manifold.*

PROOF. Let $\Omega$ be the Kähler form of a Hermitian metric on $Y$. By pulling-back, the metric on $Y$ induces a Hermitian metric on $X$ with Kähler form $f^*\Omega$. Since pull-back commutes with differentiation, the induced metric on $X$ is Kählerian and the proof is complete. ☐

We introduce now the notion of Kähler potential, which has many applications in physics, like in the study of the motion of some subatomic particles called *gauginos* (see **[6]** for instance).

THEOREM 1.4.8. *Let $Z$ be a Kähler manifold and let $\Omega$ be the Kähler form on $Z$. For every point $z \in Z$, there are a neighborhood $U \subset Z$ of $z$ and a real valued differentiable function $f$ on $U$ such that*

$$\Omega = i\,\partial\overline{\partial} f.$$

*In other words, the Kalher form of a Kähler manifold admits locally a Kähler potential.*

PROOF. The Poincaré's lemma applied to the exact real form $\Omega$ shows that there exists locally a real differential form $\omega$ of degree 1 whose differential is $\Omega$ and which can be written by $\omega = \eta + \overline{\eta}$ where $\eta$ is a bihomogeneous differential form of type (1, 0). Then we have

$$\Omega = d\omega = \partial\omega + \overline{\partial}\omega = \partial\eta + \partial\overline{\eta} + \overline{\partial}\eta + \overline{\partial}\overline{\eta}.$$

Since $\Omega$ is a differential form of type (1, 1), we must have $\partial\eta = \overline{\partial}\overline{\eta} = 0$. Thus we obtain

$$\Omega = \partial\overline{\eta} + \overline{\partial}\eta. \tag{7}$$

By Dolbeault's lemma, the equality $\overline{\partial}\overline{\eta} = 0$ implies that there is a complex valued differentiable function $\varphi$ such that $\overline{\eta} = \overline{\partial}\varphi$ and hence $\eta = \partial\overline{\varphi}$. Using this, the equality (7) becomes

$$\Omega = \partial\overline{\partial}\varphi + \overline{\partial}\partial\overline{\varphi} = \partial\overline{\partial}(\varphi - \overline{\varphi}).$$

If we define $f$ by

$$f = i(\overline{\varphi} - \varphi),$$

we obtain a real valued differentiable function such that $\Omega = i\partial\overline{\partial}f$. □

EXAMPLE 1.4.9. In $\mathbb{C}^n$, the function $f$ given by

$$f(z) = \frac{1}{2}|z|^2$$

is a Kähler potential for the standard Kähler form given in the example 1.4.6 as the following explicit computation shows:

$$\begin{aligned} i\partial\overline{\partial}f &= \frac{i}{2}\partial\overline{\partial}\sum_{\alpha=1}^{n} z_\alpha\overline{z}_\alpha \\ &= \frac{i}{2}\partial\sum_{\alpha,\beta=1}^{n}\frac{\partial}{\partial\overline{z}_\beta}(z_\alpha\overline{z}_\alpha)\,d\overline{z}_\beta \\ &= \frac{i}{2}\partial\sum_{\alpha=1}^{n} z_\alpha\,d\overline{z}_\alpha \\ &= \frac{i}{2}\sum_{\alpha=1}^{n} dz_\alpha\wedge d\overline{z}_\alpha = \Omega. \end{aligned}$$

Theorem 1.4.8 is not an equivalence but we might be interested in the converse. Suppose that $Z$ is a complex manifold and that $\Omega$ is a real differential form of type $(1,1)$ on $Z$ such that there exists on the neighborhood of every point of $Z$ a real valued differentiable function such that $\Omega = i\,\partial\overline{\partial}f$. Thus we have

$$d\Omega = i\,d\partial\overline{\partial}f = i\,\partial\partial\overline{\partial}f + i\,\overline{\partial}\partial\overline{\partial}f = -i\,\partial\overline{\partial}\overline{\partial}f = 0.$$

Hence we may ask when a differential form $\Omega$ defined on $Z$ and locally written by $i\,\partial\overline{\partial}f$ is the fundamental form of a Kählerian structure on $Z$. For this, we only have to check if the quadratic form associated to $\Omega$ is positive definite. The following result will be useful for this, and thus also to obtain some additional examples of Kählerian manifolds.

THEOREM 1.4.10. *Let $Z$ be a complex manifold of complex dimension $n$, suppose that $r$ is an integer such that $r > n$ and let $f_1, \ldots, f_r$ be holomorphic functions on an open subset $U$ of $Z$ such that*

(1) *the functions $f_1, \ldots, f_r$ does not vanish simultaneously on $U$;*

(2) *at every point $z \in U$ where $f_\alpha(z) \neq 0$ for some $1 \leq \alpha \leq r$, there are $n$ linearly independent differential forms among the $d(f_1/f_\alpha)$, ..., $d(f_r/f_\alpha)$.*

*Then the real valued differentiable function $f$ defined on $U$ by*

$$f = \ln\left(\sum_{\alpha=1}^{r} |f_\alpha|^2\right)$$

*defines a differential form $\Omega = i\,\partial\overline{\partial} f$ which is positive definite on $U$.*

PROOF. Since the functions $f_\alpha$ ($\alpha = 1, \ldots, r$) are holomorphic on $U$, we have $\partial f_\alpha = df_\alpha$, $\partial\overline{f_\alpha} = \overline{\partial} f_\alpha = 0$ and $\overline{\partial}\,\overline{f_\alpha} = d\overline{f_\alpha}$ for every $\alpha = 1, \ldots, r$. Thus we obtain

$$\Omega \;=\; i\,\partial\overline{\partial}\left(\ln\left(\sum_{\alpha=1}^{r} f_\alpha\,\overline{f_\alpha}\right)\right) = i\,\partial\left(\frac{1}{\sum_{\alpha=1}^{r}|f_\alpha|^2}\sum_{\alpha=1}^{r} f_\alpha d\overline{f_\alpha}\right).$$

Since $f_\alpha$ is holomorphic on $U$, we have $\overline{\partial}\,\overline{f_\alpha} = d\overline{f_\alpha}$ and then

$$\begin{aligned}\Omega &= i\,\partial\left(\frac{1}{\sum_{\alpha=1}^{r}|f_\alpha|^2}\sum_{\alpha=1}^{r} f_\alpha d\overline{f_\alpha}\right)\\ &= \frac{i}{2}\left(\frac{1}{\sum_{\alpha=1}^{r}|f_\alpha|^2}\right)^2\sum_{\alpha,\beta=1}^{r}\left(f_\alpha df_\beta - f_\beta df_\alpha\right)\wedge\left(\overline{f_\alpha}d\overline{f_\beta} - \overline{f_\beta}d\overline{f_\alpha}\right).\end{aligned}$$

For every $\alpha$, $\beta \in \{1, \ldots r\}$, define the 1-forms $r_{\alpha,\beta}$ by

$$r_{\alpha,\beta} = f_\alpha df_\beta - f_\beta df_\alpha.$$

The differential form

$$\left(\frac{1}{\sum_{\alpha=1}^{n} f_\alpha\,\overline{f_\alpha}}\right)^2\sum_{\alpha,\beta=1}^{n} r_{\alpha,\beta}\wedge\overline{r}_{\alpha,\beta}$$

is certainly Hermitian. Moreover, it is positive definite unless the $r_{\alpha,\beta}$ are all equal to zero. If $u \in U$, there exists an integer $1 \leq \gamma \leq n$ such that $f_\gamma(u) \neq 0$ and there are $n$ linearly independent differential forms among

$$d\left(\frac{f_1}{f_\gamma}\right),\ \ldots,\ d\left(\frac{f_r}{f_\gamma}\right).$$

We verify (reductio ad absurdum) that this implies the fact that $r_{\alpha,\beta}$ are not vanishing together and the proof is complete. □

EXAMPLE 1.4.11. Using the previous theorems, we can now give some more examples of Kählerian manifolds.

(1) Complex projective spaces are Kähler manifolds. Let $z_0, \ldots, z_n$ be coordinates of $\mathbb{C}^{n+1}$. For every $0 \leq \alpha \leq n$, consider the open set $U_\alpha = \{z \in \mathbb{C}^{n+1} \ : \ z_\alpha \neq 0\}$. The collection $(U_\alpha)_{\alpha=1,\ldots,n}$ is an open covering of $\mathbb{CP}^n$.

On the open set $U_\alpha$, we consider the holomorphic functions

$$\frac{z_0}{z_\alpha}, \frac{z_1}{z_\alpha}, \ldots, \frac{z_n}{z_\alpha}$$

which do not vanish together on $U_\alpha$ and which have linearly independant differentials at every point of $U_\alpha$. Thus by theorem 1.4.10, the function

$$f_\alpha = \ln\left(\sum_{k=0}^{n}\left|\frac{z_k}{z_\alpha}\right|^2\right)$$

defines a differential form

$$\Omega_\alpha = i\,\partial\overline{\partial}\ln\left(\sum_{\beta=0}^{n}\left|\frac{z_\beta}{z_\alpha}\right|^2\right)$$

which is positive definite on $U_\alpha$.

A direct computation shows that if $0 \leq \alpha \neq \beta \leq n$, we have

$$\Omega_\alpha - \Omega_\beta = i\,\partial\overline{\partial}\ln\left|\frac{z_\beta}{z_\alpha}\right|^2 = 0.$$

Then there is a unique, closed, and positive definite differential form $\Omega$ defined on $\mathbb{CP}^n$ whose restriction on $U_\alpha$ coincides with $\Omega_\alpha$ for every $0 \leq \alpha \leq n$. Thus $\Omega$ defines a Kählerian structure on $\mathbb{CP}^n$.

(2) Every holomorphic submanifold of $\mathbb{CP}^n$ is Kählerian because of the preceeding example and proposition 1.4.7.

The following theorem shows a necessary topological condition for a compact complex manifold to be a Kählerian manifold.

THEOREM 1.4.12. *If $Z$ is a compact Kähler manifold of complex dimension $n$, then*

$$H^{2p}(Z, \mathbb{R}) \neq 0$$

*for every $p = 0, \ldots, n$.*

PROOF. Let $\Omega = \Omega|_{\mathbb{R}}$ be the Kähler form of a Hermitian metric $h$ on $Z$ and denote by $\Omega^{(p)} = \bigwedge^p \Omega$ the $p$-fold exterior product of $\Omega$ by itself. Since $\Omega$ is closed, it follows that all $\Omega^{(p)}$ are closed too ($p = 0, \ldots, n$). Hence, the differential form $\Omega^{(p)}$ represents via de Rham's theorem, an element of $H^{2p}(Z, \mathbb{R})$. So we have to show that $\Omega^{(p)}$ is not the differential of a differential form of degree $2p-1$. Let us assume that this is the case and there exists a differential form $\eta^{2p-1}$ of degree $2p-1$ such that

$$\Omega^{(p)} = d\eta^{2p-1}.$$

We would have

$$\Omega^{(n)} = d\left(\Omega^{(n-2p)} \wedge \eta^{2p-1}\right)$$

and thus by Stoke's theorem

$$\int_Z \Omega^{(n)} = \int_Z d\left(\Omega^{(n-2p)} \wedge \eta^{2p-1}\right) = 0. \tag{8}$$

But proposition 1.4.4 and compacity of $Z$ gives that

$$\int_Z \Omega^{(n)} > 0$$

which is obviously in contradiction with (8). □

Not every compact manifold can be given a Kählerian metric. In order to illustrate this fact, let us recall a simple example of Hopf manifold. Consider the complement of the origin in the Euclidian space $\mathbb{C}^n\backslash\{0\}$ with $n \geq 2$ and let $\alpha \in \mathbb{C}\backslash\{0\}$ such that $|\alpha| \neq 1$. Then $\mathbb{Z}$ acts freely and discontinuously on $\mathbb{C}^n\backslash\{0\}$ by

$$(k,\, s) \in \mathbb{Z} \times (\mathbb{C}^n\backslash\{0\}) \mapsto \alpha^k s \in \mathbb{C}^n\backslash\{0\}.$$

The quotient $\mathcal{H}^n_\alpha = (\mathbb{C}^n\backslash\{0\})/\mathbb{Z}$ is a basic example of *Hopf manifold*. Since $n \geq 2$, the manifold $\mathbb{C}^n\backslash\{0\}$ is simply connected so it is the universal cover of $\mathcal{H}^n_\alpha$. Since the multiplication by $\alpha^k$ is biholomorphic, the quotient space $\mathcal{H}^n_\alpha$ inherits from $\mathbb{C}^n\backslash\{0\}$ the structure of a complex manifold[1]. Using polar decomposition of elements of $\mathbb{C}^n\backslash\{0\}$, we see that $\mathcal{H}^n_\alpha$ is diffeomorphic to $S^1 \times S^{2n-1}$. Taking $n = 2$, we obtain a compact complex manifold $\mathcal{H}^2_\alpha$ which is topologically equivalent to $S^1 \times S^3$. Künneth theorem show that $H^2(\mathcal{H}^2, \mathbb{R}) = 0$, thus theorem 1.4.12 gives that the Hopf manifold $\mathcal{H}^2_\alpha$ does not carry a Kähler metric.

## 1.5. Generalized flag manifolds

Before defining generalized flag manifolds, let us recall what are classical complete or partial flag manifolds. Let $V$ be a complex vector space. A *complete flag in* $V$ is a sequence of nested subspaces

$$V_1 \subset V_2 \subset \ldots \subset V_m = V$$

where $\dim^{\mathbb{C}} V_\alpha = \alpha$ for every $1 \leq \alpha \leq m$. Of course, if $\{e_1,\, \ldots,\, e_m\}$ is a basis of $V$, then a standard flag is given by

$$V_\alpha = \operatorname{span}\{e_1,\, \ldots,\, e_\alpha\},\ \alpha = 1,\, \ldots,\, m.$$

[1] A recent generalization of this example is due to Calabi and Eckmann in [**20**], where they showed that the product of two odd-dimensional spheres carries a complex structure. See also [**58**], example 2.5, chapter IX.

The variety of complete flags in $V$ has a natural structure of complex projective variety on which the group $\mathrm{SL}(V)$ acts. If we identify $V$ with $\mathbb{C}^n$ whose special linear group is the group of $n \times n$ matrices, the stabilizer of the standard flag is the group $B$ of non singular upper triangular matrices. Thus the variety of complete flags can be written as the homogeneous space $\mathrm{SL}(V)/B$.

More generally, we may consider the varieties of partial flags in $V$. Let $\mathbf{n} = (n_1, \ldots, n_p)$ be a strictly increasing sequence of natural numbers with $n_p < m = \dim^{\mathbb{C}} V$. A *partial flag of type* $\mathbf{n}$ *in* $V$ is a sequence of nested subspaces

$$V_{n_1} \subset V_{n_2} \subset \ldots \subset V_{n_p} \subset V$$

with $\dim^{\mathbb{C}} V_{n_\alpha} = n_\alpha$ for every $1 \leq \alpha \leq p$. Once again, a standard flag can be constructed from a basis $\{e_1, \ldots, e_m\}$ of $V$. Letting $n_0 = 0$, the stabilizer of the standard flag is the group of non singular block upper triangular matrices, where the dimensions of the blocks are $n_\alpha - n_{\alpha-1}$ $(\alpha = 1, \ldots p)$. If we denote by $P$ this latter group, we can describe the variety of partial flags of type $\mathbf{n}$ in $V$ by the homogeneous space $\mathrm{SL}(V)/P$. Of course, projective spaces and Grassmannians are basic examples of such varieties.

Now, we introduce a class of homogeneous spaces which generalizes partial flag varieties and which have many interesting geometrical properties, making them useful in both algebraic and differential geometry, but also in theoretical physics.

Definition 1.5.1. A *generalized flag manifold* is a compact simply connected homogeneous Kähler manifold.

Generalized flag manifolds appear in physics in a huge variety of contexts, e.g. as target manifolds for sigma models which are natural candidates for effective low-energy theories (see [**29**]) and play an important role in the current understanding of symmetry breaking (see [**3, 40**]) or as a geometric formulation of harmonic superspaces (see [**12**]).

Some works in theoretical physics about certain constant spinors lead to the discovery of some relations between the spectra of wave generators of different spins in Einstein-Kähler spaces (see [**71**]). Since every generalized flag manifold can be endowed with an explicit Kähler-Einstein metric with Einstein constant equal to 1 (see [**5**]), this leads to some significant simplifications on the computation of the Ricci tensor and is closely related to the Einstein's so-called cosmological constant (see [**9**]).

Proposition 1.5.2. *If $P$ is a parabolic subgroup in $G$ then $G/P$ is a Kählerian complex manifold. Conversely, every compact simply connected $G$-homogeneous Kähler manifold is of the form $G/P$ with $P$ parabolic in $G$.*

PROOF. Let us suppose that $P$ is a parabolic subgroup of $G$. By definition, the complex manifold $G/P$ is a complex projective variety and hence is Kählerian. The last statement is proven in [**88**]. □

## 1.6. Bott-Borel-Weil theorem

In the University of Princeton, A. BOREL and A. WEIL were working together during the academic year of 1953. They discovered an explicit construction of the irreducible representations of compact connected Lie groups. The realisations given for such representations are of geometric nature, in terms of spaces of holomorphic sections of holomorphic line bundles. The results of A. BOREL and A. WEIL were not published immediately but there are some notes of this subject written by A. BOREL in 1954 (see [**13**]) and J.-P. SERRE lectured on the so-called Borel-Weil theorem in the Séminaire Bourbaki n°100 of May 1954 (see [**81**]). Independently of A. BOREL and A. WEIL, J. TITS made the same discovery one year later (see [**85**] pp.112-113).

Precisely, here is the Borel-Weil theorem. Suppose that $G$ is a semisimple complex Lie group and $B$ is a Borel subgroup of $G$. If $\rho : B \to \mathfrak{gl}(V)$ is a representation of $B$ on a finite dimensional complex vector space $V$, it induces a $G$-homogeneous vector bundle on $G/B$, namely $\pi : G \times_B V \to G/B$, where $G \times_B V$ is the quotient of $G \times V$ by the action of $B$ defined by

$$b \cdot (g, v) = (gb^{-1}, \rho(b)v),\ b \in B,\ g \in G,\ v \in V.$$

A holomorphic section of the vector bundle $\pi$ may be viewed as a $V$-valued holomorphic function $f$ defined on $G$ satisfying the equivariance condition:

$$f(gb) = \rho(b^{-1})f(g)$$

for all $g \in G$ and $b \in B$. The sheaf of holomorphic sections of $\pi$ is denoted by $\mathcal{O}_{G/B}(V)$. In the case where $V$ is the irreducible finite dimensional complex representation of $B$ with highest weight $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$, then we will simply denote $\mathcal{O}_{G/B}(V)$ by $\mathcal{O}_{G/B}(\lambda)$.

The historical Borel-Weil theorem states that if $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ is dominant for $\mathfrak{g}$, the space of global sections $\Gamma(G/B, \mathcal{O}_{G/B}(\lambda))$ is an irreducible representation space of $G$ with highest weight $\lambda$.

Later, R. BOTT generalized in [**16**] the construction of the Borel-Weil theorem to allow more liberty on $\lambda$ since in Bott's study, the weight $\lambda$ has not to be dominant for $\mathfrak{g}$. Two cases appeared, whether $\lambda + \delta$ is on some wall or not. If $\lambda$ is affinely singular, then $H^k(G/B, \mathcal{O}_{G/B}(\lambda))$ vanishes for every $k \in \mathbb{N}$. Otherwise, if we denote by $w$ the unique element of the Weyl group of $\mathfrak{g}$ such that $w \cdot \lambda$ is dominant for $\mathfrak{g}$, we have

$$H^{\ell(w)}(G/B, \mathcal{O}_{G/B}(\lambda)) = \Gamma(G/B, \mathcal{O}_{G/B}(w \cdot \lambda))$$

and all other cohomology degree vanishes.

A parabolic version of the Bott-Borel-Weil theorem can be stated when we consider parabolic subgroups instead of Borel subgroups. Suppose that $Q$ is a parabolic subgroup of $G$ and $\lambda \in \mathfrak{h}^*_{\mathbb{Z}}$ is dominant for $Q$. As in the Borel case, we denote by $\mathcal{O}_{G/Q}(\lambda)$ the sheaf of germs of holomorphic sections of the $G$-homogeneous vector bundle $G \times_Q V_Q(\lambda) \to G/Q$ where $V_Q(\lambda)$ is a finite dimensional irreducible complex representation space of $Q$ with highest weight $\lambda$. Two cases can arise for such $\lambda$ whether it is affinely singular for $G$ or not.

THEOREM 1.6.1. *In the situation described above, we have*

$$H^k(G/Q, \mathcal{O}_{G/Q}(\lambda)) = 0$$

*for every $k \in \mathbb{N}$ if $\lambda$ is affinely singular for $G$. Otherwise, we denote by $w$ the unique element of $W_{\mathfrak{g}}$ such that $w \cdot \lambda$ is dominant for $G$ and all cohomology spaces vanishes but*

$$H^{\ell(w)}(G/Q, \mathcal{O}_{G/Q}(\lambda)) = V_G(w \cdot \lambda)$$

*as $G$-modules.*

For a proof of this result, the interested reader can refer to **[7, 61]** or **[84]** chapter 16.

CHAPTER 2

# Equivariant $\mathcal{O}$-modules and quasi-equivariant $\mathcal{D}$-modules

This chapter recall some other main tools which are used in this thesis. The full understanding of these results cannot be achieved without having some good knowledge in sheaf and $\mathcal{D}$-modules theory and once again, it would be inconceivable to recall here what would be needed. Thus we assume that the reader is familiar with the basics of category theory, homological algebra and sheaf theory. Moreover, he should also have a good knowledge of the category of $\mathcal{D}$-modules and the related fundamental operations, which will not be recalled here in the non-equivariant setting. This exposition has been elaborated using the following reference books : [**10, 11, 37, 50, 51, 62**] but many useful texts and additional informations about these tools are shown in the bibliography, like [**4, 27, 51, 62, 65, 67, 74, 80**] for category theory, [**51, 74, 75**] for homological algebra, [**11, 17, 18, 30, 32, 37, 50, 51, 73, 75**] for sheaf theory, and [**10, 11, 39, 46, 78, 79**] for $\mathcal{D}$-modules.

This aim of this chapter is multiple. On one hand, the Penrose transformation has been studied in the framework of $\mathcal{D}$-modules by many authors such as A. D’Agnolo, P. Schapira and C. Marastoni) (see [**24, 22, 23, 25**]) and, more recently, in the quasi-equivariant setting by C. Marastoni and T. Tanisaki (see [**64**]). On the other hand, M. Kashiwara has established in [**47**] a correspondance between quasi-equivariant $\mathcal{D}$-modules on generalized flag varieties and some modules which are simultaneously endowed with a Lie group action and a Lie algebra action compatible in a way. The first aim of this thesis is to determine the algebraic analogs of the inverse and direct image functors for $\mathcal{D}$-modules. Once this task is complete, we will deduce from an algebraic version of the classical Bott-Borel-Weil the algebraic variant of the $\mathcal{D}$-modules Penrose transformation. Finally, we will explain a method to analyze the image by this algebraic Penrose transformation of a generalized Verma module associated with a given dominant and integral weight.

The gist of the chapter 2 is the following. First, we will recall the notion of equivariance for $\mathcal{O}$-modules and quasi-equivariance for algebraic $\mathcal{D}$-modules, including the construction of the associated derived categories. Some results about the fundamental operations for $\mathcal{D}$-modules are established but we will limit our presentation to the results and constructions which will be useful at the core of this dissertation. Once the quasi-equivariance scene is set for $\mathcal{D}$-modules, we will recall the correspondence according to M. KASHIWARA between quasi-equivariant $\mathcal{D}$- modules and representations of Lie groups. Then, we will define the fundamental operations of internal and exterior tensor product in the category of quasi-equivariant $\mathcal{D}$-modules and describe the semi-outer Hom functor. We will also define the inverse and direct image functors for quasi-equivariant $\mathcal{D}$-modules and deduce that in this setting, similarly to the non-equivariant case, those two functors are adjoint one to each other up to a shift. We explain the well-known equivalence of categories according to M. KASHIWARA between some categories of quasi-equivariant $\mathcal{D}$-modules and some categories of Lie group representations.

## 2.1. Equivariant $\mathcal{O}$-modules survey

This section is devoted to the introduction of the category of equivariant $\mathcal{O}$-modules, of the associated derived categories and of some of basic properties which will be useful for the core of the subject of this dissertation. We will assume that $G$ is a complex algebraic group and $Z$ a smooth algebraic variety endowed with an algebraic action of $G$. Such algebraic varieties will be called *algebraic $G$-manifolds*. First, we will define the notion of $G$-equivariance for $\mathcal{O}_Z$-modules. Then, we will explain how the derived categories of $G$-equivariant $\mathcal{O}_Z$-modules are constructed, using an equivariant embedding technique which was originally used by H. SUMIHIRO in [**83**]. However, since the original proof is quite long, we will prove this result with the help of an alternative and elegant approach proposed by F. KNOP, H. KRAFT and T. VUST (see [**57**]). Finally, we explain the correspondence of categories given by D. MUMFORD in [**66**] between the category of $G$-equivariant $\mathcal{O}_{G/H}$-modules and the category of $H$-modules when $H$ is a closed algebraic Lie subgroup of $G$.

Let us begin with the motivation of the notion of equivariance for $\mathcal{O}$-modules. There are two natural maps

$$G \times Z \underset{\mu}{\overset{\mathrm{pr}}{\rightrightarrows}} Z$$

where $\mu : G \times Z \to Z$ is the action morphism and $\mathrm{pr} : G \times Z \to Z$ is the canonical projection. When there is no risk of confusion, we use the following notation $gz = \mu(g, z)$ for $g \in G$ and $z \in Z$.

An invariant function $f : Z \to \mathbb{C}$ is a function such that

$$f(gz) = f(z)$$

for any $g \in G$ and any $z \in Z$. Using pullbacks, we see that for $g \in G$ and $z \in Z$,

$$(\mu^* f)(g, z) = f(\mu(g, z)) = f(gz) = f(z) = f(\mathrm{pr}(g, z)) = (\mathrm{pr}^* f)(g, z)$$

and we see that the invariance means that $\mu^* f = \mathrm{pr}^* f$.

Define the morphisms $q_i \; : \; G \times G \times Z \to G \times Z$ ($i = 1, 2, 3$) by $q_1(g_1, g_2, z) = (g_1, g_2 z)$, $q_2(g_1, g_2, z) = (g_1 g_2, z)$ and $q_3(g_1, g_2, z) = (g_2, z)$ for $g_1, g_2 \in G$ and $z \in Z$. We check that we have the following simplicial diagram

$$G \times G \times Z \underset{q_3}{\overset{q_1}{\underset{q_2}{\rightrightarrows}}} G \times Z \underset{\mathrm{pr}}{\overset{\mu}{\rightrightarrows}} Z,$$

ie. that we have $\mu \circ q_1 = \mu \circ q_2$, $\mathrm{pr} \circ q_2 = \mathrm{pr} \circ q_3$ and $\mathrm{pr} \circ q_1 = \mu \circ q_3$.

Definition 2.1.1. A *$G$-equivariant $\mathcal{O}_Z$-module* is a $\mathcal{O}_Z$-module $\mathcal{F}$ endowed with a $\mathcal{O}_{G\times Z}$-linear isomorphism $\beta_{\mathcal{F}} \; : \; \mu^*\mathcal{F} \to \mathrm{pr}^* \mathcal{F}$ such that the following diagram commutes:

$$\begin{array}{ccc} q_2^*\mu^*\mathcal{F} & \xrightarrow{\quad q_2^*\beta_{\mathcal{F}} \quad} & q_2^* \mathrm{pr}^* \mathcal{F} \\ \wr\downarrow & & \downarrow\wr \\ q_1^*\mu^*\mathcal{F} \xrightarrow[q_1^*\beta_{\mathcal{F}}]{} q_1^* \mathrm{pr}^* \mathcal{F} & \cong & q_3^*\mu^*\mathcal{F} \xrightarrow[q_3^*\beta_{\mathcal{F}}]{} q_3^* \mathrm{pr}^* \mathcal{F}. \end{array} \tag{9}$$

Example 2.1.2. As a $\mathcal{O}_Z$-module over itself, $\mathcal{O}_Z$ is $G$-equivariant: the isomorphism $\beta_{\mathcal{O}_Z}$ is given by the composition of the canonical isomorphisms $\mu^*\mathcal{O}_Z \xrightarrow{\sim} \mathcal{O}_{G\times Z} \xrightarrow{\sim} \mathrm{pr}^* \mathcal{O}_Z$.

Example 2.1.3. Suppose that $\mathcal{E}$ is a locally free $\mathcal{O}_Z$-module associated to a vector bundle $\pi \; : \; E \to Z$. Now suppose that $\beta \; : \; \mu^*\mathcal{E} \to \mathrm{pr}^* \mathcal{E}$ is a $\mathcal{O}_{G\times Z}$-linear isomorphism making $\mathcal{E}$ a $G$-equivariant $\mathcal{O}_Z$-module. Giving $\beta$ amounts to giving a vector bundle isomorphism $\mu^{-1}E \to \mathrm{pr}^{-1} E$. Observe that this isomorphism determines and is determined by the data of a map $\Phi \; : G \times E \to E$ where, for $g \in G$, the map $\Phi \; : \{g\} \times E \to E$ is given by restricting $\beta$ to $\{g\} \times Z$. Rewriting the commutating relations (9) in terms of the map $\Phi$, we obtain that

$$\Phi(g_2, \Phi(g_1, f)) = \Phi(g_1 g_2, f)$$

and

$$\Phi(e, f) = f$$

for all $g_1, g_2 \in G$ and all $f \in \mathcal{E}$. These equalities means that the map $\Phi$ gives an action of $G$ on $E$. We can thus conclude that giving a $G$-equivariant structure on a vector bundle $\mathcal{E}$ is the same as giving an action of $G$ on the total space $E$ of $\mathcal{E}$ such that

(1) the projection $\pi \; : \; E \to Z$ commutes with action of $G$ and in particular, any $g \in G$ sends the fiber $E_z$ over $z \in Z$ to $E_{gz}$;
(2) for any $z \in Z$ and $g \in G$, the map $\Phi(g, \cdot) \; : \; E_z \to E_{gz}$ is a linear map of vector spaces.

Recall that a sheaf of $\mathcal{O}_Z$-modules $\mathcal{F}$ is *quasi-coherent* if it has a local presentation, i.e. if there exists an open cover $U_i$ ($i \in I$) of $Z$ and exact sequences

$$\mathcal{O}^{(J)}|_{U_i} \to \mathcal{O}^{(J')}|_{U_i} \to \mathcal{F}|_{U_i} \to 0$$

where the first two terms are restrictions on $U_i$ of (possibly infinite) direct sums of copies of the structure sheaf.

Definition 2.1.4. We denote by $\mathrm{Mod}(\mathcal{O}_Z)$ the category of $\mathcal{O}_Z$-modules, by $\mathrm{Mod}_G(\mathcal{O}_Z)$ the abelian category where objects are $G$-equivariant $\mathcal{O}_Z$-modules which are quasi-coherent as $\mathcal{O}_Z$-modules and the morphisms between two objects are the morphisms of $\mathcal{O}_Z$-modules which are $G$-equivariant. We also denote by $\mathrm{Mod}_{G,\,\mathrm{coh}}(\mathcal{O}_Z)$ the full subcategory of $\mathrm{Mod}_G(\mathcal{O}_Z)$ consisting of coherent objects.

Lemma 2.1.5. *Let $\mathcal{F}$ be a $G$-equivariant coherent sheaf on $Z$. The space $\Gamma(Z,\,\mathcal{F})$ of global sections of $\mathcal{F}$ has a natural structure of an algebraic $G$-module.*

Proof. We first see that the diagram

$$\begin{array}{ccc} G \times Z & \xrightarrow{\mathrm{pr}} & Z \\ {\scriptstyle p_G}\downarrow & & \downarrow \\ G & \longrightarrow & \{\mathrm{pt}\} \end{array}$$

is a cartesian square. Thus since $G$ is affine, we obtain a a canonical isomorphism

$$\Gamma(G \times Z,\, \mathrm{pr}^* \mathcal{F}) \cong \mathcal{O}_G \otimes \Gamma(Z,\, \mathcal{F}).$$

Using the action map $\mu$ and the isomorphism $\beta_F$ given with the equivariant structure of $\mathcal{F}$, we obtain a composition of linear maps

$$\Gamma(Z,\, \mathcal{F}) \xrightarrow{\mu^*} \Gamma(G \times Z,\, \mu^* \mathcal{F}) \xrightarrow{\beta_{\mathcal{F}}} \Gamma(G \times Z,\, \mathrm{pr}^* \mathcal{F}) = \mathcal{O}_G \otimes \Gamma(Z,\, \mathcal{F}).$$

It remains to check by explicit computations that this composition of maps gives to $\Gamma(Z,\, \mathcal{F})$ the structure of an algebraic $G$-module. $\square$

Moreover, we can show that there are many $G$-equivariant line bundles on a $G$-variety, which will be useful for the demonstration of the projective embedding theorem 2.1.7.

Theorem 2.1.6. *If $L$ is an algebraic line bundle on $Z$, then there exists a positive integer $n$ such that the line bundle $L^{\otimes n}$ admits a $G$-equivariant structure (which is not unique in general).*

PROOF. A proof of this result can be found in [**56**] (proposition 2.4 p.67). □

**2.1.1. Derived categories of equivariant $\mathcal{O}$-modules.** In our discussion so far, no particular assumptions were imposed to the complex algebraic group $G$ neither to the algebraic $G$-manifold $Z$.

We suppose now that $G$ is an affine algebraic group and $Z$ is a quasi-projective $G$-manifold (i.e. isomorphic to a locally closed subvariety of a projective space). These assumptions will be satisfied later on for reductive $G$ and generalized flag manifold $Z$.

Under these assumptions on $G$ and $Z$, we can state a famous equivariant embedding result originally given in 1974 by H. SUMIHIRO (see [**83**]). The proof given here is not the original one but is based on an elegant approach of F. KNOP, H. KRAFT and T. VUST (see [**57**]). Note that the special case when $Z$ is projective was proved in 1966 by T. KAMBAYASHI [**45**].

PROPOSITION 2.1.7. *There exists a finite dimensional vector space $V$, an algebraic group homomorphism $\rho : G \to \mathrm{GL}(V)$ and an embedding $i : Z \to \mathbb{P}(V)$ which satisfies the following equivariance condition:*

$$i(gz) = \rho(g) \cdot i(z) \quad \text{for all } g \in G,\ z \in Z.$$

*Here the dot on the right side of the equality stands for the standard action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$. In other words, there is a $G$-equivariant open embedding from $Z$ into a projective $G$-manifold.*

Before sketching the proof of this result, recall that a line bundle $\mathcal{L}$ on $Z$ is called *ample* if, for any coherent sheaf $\mathcal{F}$ on $Z$, there exists an integer $n$ (depending on $\mathcal{F}$) such that the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its global sections i.e. there exists a set of global sections whose restrictions to each stalk form a set of generators.

EXAMPLE 2.1.8. Let $Z = \mathbb{CP}(n)$ for some integer $n$. For $d \in \mathbb{Z}$, denote by $\mathcal{O}(d)$ the sheaf of homogeneous functions of degree $d$ on $Z$. We have $\mathcal{O}(0) = \mathcal{O}_Z$ and the multiplication gives natural homomorphisms

$$\mathcal{O}(d_1) \otimes_{\mathcal{O}_Z} \mathcal{O}(d_2) \to \mathcal{O}(d_1 + d_2)$$

which are isomorphisms. Each sheaf $\mathcal{O}(d)$ ($d \in \mathbb{Z}$) is invertible and it is well-known (see [**53**] lemma 5.5.4 p.65 for instance) that any invertible sheaf on $\mathbb{CP}(n)$ is isomorphic to some $\mathcal{O}(d)$ where $d$ is uniquely determined when $n > 0$. Moreover, since $\mathcal{O}(1)$ is ample, we obtain that $\mathcal{O}(k)$ is ample for every $k > 0$. On the other hand, since $\mathcal{O}(k)$ has no global sections for $k < 0$, the sheaf $\mathcal{O}(k)$ cannot be ample when $k < 0$. So on $Z = \mathbb{CP}^n$, we have that $\mathcal{O}(k)$ is ample if and only if $k > 0$.

PROOF OF PROPOSITION 2.1.7. Since $Z$ is quasi-projective, there is a projective (but non-equivariant) embedding of $Z$ as a dense Zariski open subset of a projective variety $\overline{Z}$. Denote by $i : \overline{Z} \to \mathbb{CP}^n$ an embedding of the projective variety $\overline{Z}$ into $\mathbb{CP}^n$.

Denote by $\mathcal{L}$ the ample line bundle on $\overline{Z}$ induced from the sheaf $\mathcal{O}(1)$ by means of the embedding $i : \overline{Z} \to \mathbb{CP}^n$. The theorem 2.1.6 gives the existence of a positive integer $n$ such that $\mathcal{L}|_Z^{\otimes n}$ admit a $G$-equivariant structure. From now on, replacing $\mathcal{L}$ by $\mathcal{L}^{\otimes n}$, we will assume that the sheaf $\mathcal{L}|_Z$ is itself $G$-equivariant.

For each $z \in \overline{Z}$, let $H(z)$ be the hyperplane in the finite dimensional vector space $\Gamma(\overline{Z}, \mathcal{L})$ consisting of all sections vanishing at $z$. Then it is well known that the map

$$z \in \overline{Z} \mapsto H(z) \subset \Gamma(\overline{Z}, \mathcal{L})$$

induces a projective embedding $j : \overline{Z} \to \mathbb{P}(\Gamma(\overline{Z}, \mathcal{L}))$.

Finally, consider a canonical embedding

$$\Gamma(\overline{Z}, \mathcal{L}) \subset \Gamma(Z, \mathcal{L}).$$

Lemma 2.1.5 gives that the space $\Gamma(Z, \mathcal{L})$ has a natural structure of algebraic $G$-module. One has to remark that $\Gamma(\overline{Z}, \mathcal{L})$ is finite dimensional but is not necessarily $G$-stable. But since the action of $G$ is algebraic, there is a finite dimensional $G$-stable subspace $V$ in $\Gamma(Z, \mathcal{L})$ containing $\Gamma(\overline{Z}, \mathcal{L})$.

It remains to check that the map which assings to any $z \in Z$ the hyperplane in $V$ consisting of sections of $\mathcal{L}$ on $Z$ vanishing at $z$ yields the desired $G$-equivariant embedding. □

PROPOSITION 2.1.9. *Any $G$-equivariant coherent $\mathcal{O}_Z$-module $\mathcal{F}$ on $Z$ is a quotient of a $G$-equivariant locally free $\mathcal{O}_Z$-module.*

PROOF. Proceeding as in the proof of 2.1.7, we obtain a projective variety $\overline{Z}$ containing $Z$ and an ample line bundle $\mathcal{L}$ on $\overline{Z}$. Also, let us extend the sheaf $\mathcal{F}$ to a coherent (but not necessarily equivariant) sheaf $\overline{\mathcal{F}}$ on $\overline{Z}$ such that there is an integer $n$ large enough so that $\overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n}$ is generated by a finite number of global sections $s_0, \ldots, s_p$ ($p \in \mathbb{N}_0$) on $\overline{Z}$ (see **[15]** for details). Since $s_0, \ldots, s_p$ generate $\overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n}$ on $\overline{Z}$, these sections generate also $\mathcal{F} \otimes \mathcal{L}^{\otimes n}|_Z$ on $Z$. Moreover, since $\mathcal{L}^{\otimes n}|_Z$ is a line bundle on $Z$, the theorem 2.1.6 allow us to assume that it has a $G$-equivariant structure. Like in the proof of the proposition 2.1.7, we obtain a $G$-stable and finite dimensional substace $V$ of $\Gamma(Z, \mathcal{F} \otimes \mathcal{L}^{\otimes n}|_Z)$ which contains $s_0|_Z, \ldots, s_p|_Z$. Finally, since $\overline{\mathcal{F}} \otimes \mathcal{L}^{\otimes n}$ is generated by $s_0, \ldots, s_p$, the natural map

$$V \otimes (\mathcal{L}^*|_Z)^{\otimes n} \to \mathcal{F}$$

is surjective and gives the desired quotient. □

Although not explicitely written in H. SUMIHIRO's paper [**83**], we have the following:

COROLLARY 2.1.10. *Any $G$-equivariant coherent $\mathcal{O}_Z$-module $\mathcal{F}$ on $Z$ has a finite locally free $G$-equivariant resolution.*

PROOF. The proposition 2.1.9 gives that there is a short exact sequence

$$0 \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$$

where $\mathcal{F}_1$ is a $G$-equivariant locally free $\mathcal{O}_Z$-module and $\mathcal{F}_2$ is the kernel of the morphism $\mathcal{F}_1 \to \mathcal{F}$. By iterating the same proposition, we construct resolutions

$$\ldots \to \mathcal{F}_{n+1} \to \mathcal{F}_n \to \cdots \to \mathcal{F}_2 \to \mathcal{F}_1 \to \mathcal{F} \to 0$$

by locally free coherent $G$-equivariant $\mathcal{O}_Z$-modules. The Hilbert's Syzygy theorem states that if $\dim Z = N$, then $\ker(\mathcal{F}_N \to \mathcal{F}_{N-1})$ is locally free and it insures that this resolution can be made finite by taking $\ker(\mathcal{F}_N \to \mathcal{F}_{N-1})$ to be its last term. □

PROPOSITION 2.1.11. *There is an ample invertible $G$-equivariant $\mathcal{O}_Z$-module.*

PROOF. Since any quasi-projective manifold admits an ample line bundle by pulling back $\mathcal{O}(1)$ using a projective embedding, this result follows from theorem 2.1.6. □

PROPOSITION 2.1.12. *Let $\mathcal{L}$ be an invertible and ample $G$-equivariant $\mathcal{O}_Z$-module. If $\mathcal{F}$ is a coherent $G$-equivariant $\mathcal{O}_Z$-module, there exists a sufficiently large integer $n$, a finite dimensional $G$-module $V$ and a $G$-equivariant surjective homomorphism*

$$V \otimes \mathcal{L}^{\otimes -n} \to \mathcal{F}.$$

PROOF. Since $\mathcal{L}$ is an invertible ample $G$-equivariant $\mathcal{O}_Z$-module and $\mathcal{F}$ is coherent, there is an integer $n$ such that $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by global sections. Let $V$ be a finite dimensional $G$-submodule of $\Gamma(Z, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ such that

$$V \otimes \mathcal{O}_Z \to \mathcal{F} \otimes \mathcal{L}^{\otimes n} \to 0$$

is an exact sequence in $\mathrm{Mod}_G(\mathcal{O}_Z)$. The conclusion follows since the functor $- \otimes \mathcal{L}^{\otimes -n}$ is right exact. □

This proposition implies the following exactitude criterion:

COROLLARY 2.1.13. *Let $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ be a sequence in* $\mathrm{Mod}_G(\mathcal{O}_Z)$. *If the sequence*

$$\mathrm{Hom}_{\mathrm{Mod}(O_Z,G)}(\mathcal{E}, \mathcal{F}') \to \mathrm{Hom}_{\mathrm{Mod}(O_Z,G)}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathrm{Mod}(O_Z,G)}(\mathcal{E}, \mathcal{F}'')$$

*is exact for any locally free $G$-equivariant $\mathcal{O}_Z$-module $\mathcal{E}$ of finite rank, then the sequence $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact.*

COROLLARY 2.1.14. *Let $\mathcal{I}$ be an injective object of* $\mathrm{Mod}_G(\mathcal{O}_Z)$. *The functor*

$$\mathrm{Mod}_{G,\, coh}(\mathcal{O}_Z)^{op} \to \mathrm{Mod}_G(\mathcal{O}_Z) \ : \ \mathcal{F} \mapsto \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I})$$

*is exact.*

PROOF. If $\mathcal{E}, \mathcal{F} \in \mathrm{Mod}_{G,\,\mathrm{coh}}(\mathcal{O}_Z)$ and $\mathcal{E}$ is locally free, we have

$$\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{E}, \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I})) \cong \mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{E} \otimes_{\mathcal{O}_Z} \mathcal{F}, \mathcal{I}).$$

Since $\mathcal{I}$ is injective in $\mathrm{Mod}_G(\mathcal{O}_Z)$ and $\mathcal{E}$ is locally free, the functor

$$\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{E} \otimes_{\mathcal{O}_Z} -, \mathcal{I}) \ : \ \mathrm{Mod}_G(\mathcal{O}_Z) \to \mathrm{Mod}_G(\mathcal{O}_Z)$$

is exact hence the functor

$$\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{E}, \mathcal{H}om_{\mathcal{O}_Z}(-, \mathcal{I})) \ : \ \mathrm{Mod}_G(\mathcal{O}_Z) \to \mathrm{Mod}_G(\mathcal{O}_Z)$$

is exact too and the conclusion follows from corollary 2.1.13. □

PROPOSITION 2.1.15. *Let $\mathcal{I}$ be an injective object of* $\mathrm{Mod}_G(\mathcal{O}_Z)$. *For any $\mathcal{F} \in \mathrm{Mod}_{G,\, coh}(\mathcal{O}_Z)$, we have*

$$\mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I}) = 0$$

*for every $k \in \mathbb{N}_0$.*

PROOF. We begin with the global case, which will be proved by induction on $k \in \mathbb{N}_0$. First, let us deal with the projective case and suppose that $Z$ is a projective manifold. Since

$$\mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I}) = \varinjlim_{\mathcal{E}} \mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{E})$$

where $\mathcal{E}$ ranges over the set of coherent $G$-equivariant $\mathcal{O}_Z$-submodules of $\mathcal{I}$, the proof reduces to show that the map

$$\mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I})$$

vanishes for $k \in \mathbb{N}_0$ and such $\mathcal{E}$. For sufficiently large $n$, lemma 2.1.12 gives that there is a finite dimensional $G$-module $V$ and a $G$-equivariant surjective homomorphism

$$V \otimes \mathcal{L}^{\otimes -n} \to \mathcal{F}.$$

If we denote by $\mathcal{F}'$ the kernel of this morphism, we obtain the following exact sequence
$$0 \to \mathcal{F}' \to V \otimes \mathcal{L}^{\otimes -n} \to \mathcal{F} \to 0.$$
Since we can assume that $n$ is sufficiently large to have
$$H^m(Z, \mathcal{E} \otimes \mathcal{L}^{\otimes n}) = 0$$
for any $m \in \mathbb{N}_0$, we can assume that
$$\mathrm{Ext}^m_{\mathcal{O}_Z}(V \otimes \mathcal{L}^{\otimes -n}, \mathcal{E}) \cong V^* \otimes H^m(Z, \mathcal{E} \otimes \mathcal{L}^{\otimes n}) = 0$$
for $m \in \mathbb{N}_0$. Thus for $k \in \mathbb{N}_0$, we obtain a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathrm{Ext}^{k-1}_{\mathcal{O}_X}(V \otimes \mathcal{L}^{\otimes -n}, \mathcal{E}) & \longrightarrow & \mathrm{Ext}^{k-1}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{E}) & \xrightarrow{\beta'} & \mathrm{Ext}^{k}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{E}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Ext}^{k-1}_{\mathcal{O}_X}(V \otimes \mathcal{L}^{\otimes -n}, \mathcal{I}) & \xrightarrow[\alpha]{} & \mathrm{Ext}^{k-1}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}) & \xrightarrow[\beta]{} & \mathrm{Ext}^{k}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) & &
\end{array}$$

For $k = 1$, the injectivity of $\mathcal{I}$ gives that the map $\alpha$ is surjective. Since the second row of the diagram is exact, surjectivity of $\alpha$ implies that $\beta = 0$, and the surjectivity of $\beta'$ gives that the map
$$\mathrm{Ext}^1_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^1_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I})$$
vanishes. Proceed now by induction on $k > 1$. Since the induction hypothesis allow us to assume that $\mathrm{Ext}^{k-1}_{\mathcal{O}_X}(\mathcal{F}', \mathcal{I}) = 0$, we get that the map $\alpha$ is again surjective and the same reasoning allow us to conclude that the map
$$\mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{E}) \to \mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I})$$
vanishes too. Hence the global case is proven in the projective setting.

Now assume that $Z$ is quasi-projective. Using the proposition 2.1.7, there is a $G$-equivariant open embedding of $Z$ into a projective $G$-manifold $\overline{Z}$. Let us denote this embedding by $i$. If $\mathcal{F}'$ is a $G$-equivariant $\mathcal{O}_{\overline{Z}}$-module, we know the following adjunction

(10) $$\mathrm{Hom}_{\mathcal{O}_{\overline{Z}}}(\mathcal{F}', i_*\mathcal{I}) \cong \mathrm{Hom}_{\mathcal{O}_Z}(i^{-1}\mathcal{F}', \mathcal{I}).$$

By injectivity of $\mathcal{I}$ and exactness of $i^{-1}$, the isomorphism (10) shows that the functor $\mathrm{Hom}_{\mathcal{O}_{\overline{Z}}}(-, i_*\mathcal{I})$ i.e. that $i_*\mathcal{I}$ is an injective object of $\mathrm{Mod}_G(\mathcal{O}_{\overline{Z}})$. If we denote by $I$ the defining ideal of $\overline{Z}\backslash Z$, then $I$ is a coherent $G$-equivariant ideal of $\mathcal{O}_{\overline{Z}}$. Take a coherent $G$-equivariant $\mathcal{O}_{\overline{Z}}$-module $\overline{\mathcal{F}}$ such that its restriction to $Z$ coincides with $\mathcal{F}$. A. Grothendieck gives in SGA2 (see **[33, 34]**) that
$$\mathrm{Ext}^k_{\mathcal{O}_Z}(\mathcal{F}, \mathcal{I}) \cong \varinjlim_n \mathrm{Ext}^k_{\mathcal{O}_{\overline{Z}}}(\overline{F} \otimes_{\mathcal{O}_Z} I^n, i_*\mathcal{I}).$$

This concludes the proof since it remains to apply the global case in the projective setting. □

Finally, we state the following proposition which will be useful for the definition of direct image functors in the context of quasi-$G$-equivariant $\mathcal{D}$-modules.

PROPOSITION 2.1.16. *Let $f : X \to Y$ be a $G$-equivariant morphism between quasi-projective algebraic $G$-manifolds and $\mathcal{F} \in \mathrm{Mod}_{G,\,coh}(\mathcal{O}_X)$. If $\mathcal{I}$ is an injective object of $\mathrm{Mod}_G(\mathcal{O}_X)$ then*

$$R^k f_*(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\, I)) = 0$$

*for $k \in \mathbb{N}_0$.*

PROOF. Once again, let us deal with the projective case first. First remark that for any locally free $G$-equivariant and coherent $\mathcal{O}_Y$-module $\mathcal{E}$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_X)}(f^*\mathcal{E}\otimes_{\mathcal{O}_X}\mathcal{F},\mathcal{I}) &\cong \mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_X)}(f^*\mathcal{E},\, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}))\\ &\cong \mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Y)}(\mathcal{E},\, f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}))\,.\end{aligned}$$

For any locally free $G$-equivariant and coherent $\mathcal{O}_Y$-module $\mathcal{E}$, since the functor $f^*\mathcal{E}\otimes_{\mathcal{O}_X} -$ is exact and since $\mathcal{I}$ is injective, we get that the functor

$$\mathrm{Mod}_{G,\,\mathrm{coh}}(\mathcal{O}_X) \to \mathrm{Mod}_G(\mathcal{O}_Y) \;:\; \mathcal{F} \mapsto \mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Y)}(\mathcal{E},\, f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}))$$

is exact. Thus the exactness criterion 2.1.13 gives that the functor

$$\mathrm{Mod}_{G,\,\mathrm{coh}}(\mathcal{O}_X) \to \mathrm{Mod}_G(\mathcal{O}_Y) \;:\; \mathcal{F} \mapsto f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I})$$

is exact. The end of the proof is similar to the proof of proposition 2.1.15.

The general case where $X$ and $Y$ are quasi-projective is reduced to the projective case exactly like in the proposition 2.1.15. The $G$-equivariant morphism $f : X \to Y$ can be embedded in a $G$-equivariant morphism $f : \overline{X} \to \overline{Y}$ between two projective algebraic $G$-manifolds $\overline{X}$ and $\overline{Y}$. Denote by $I$ the defining ideal of $\overline{X}\backslash X$ and by $i : X \to \overline{X}$ the open $G$-equivariant embedding. If we extend $\mathcal{F}$ to a coherent $G$-equivariant $\mathcal{O}_{\overline{X}}$-module $\overline{F}$, we have again

$$R^k f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{I}) \cong \varinjlim_n R^k\overline{f}_*\mathcal{H}om_{\mathcal{O}_{\overline{X}}}(\overline{\mathcal{F}}\otimes I^n,\, i_*\mathcal{I})|_Y.$$

Hence the proof of the quasi-projective case reduces to the proof for the projective case. □

**2.1.2. Equivariant $\mathcal{O}$-modules and representation theory.** In this section, we explain a close link between equivariant $\mathcal{O}$-modules and representations of some Lie groups in terms of an equivalence of categories. This equivalence of categories is stated when the action of $G$ on $Z$ is transitive and this hypothesis will be assumed soon. But for now, simply assume that $G$ be a complex algebraic group with $\mathfrak{g}$ as associated complex Lie algebra, and that $Z$ is an algebraic $G$-manifold. Recall from section 2.1 that $\mu : G \times Z \to Z$ is the action morphism, that $i : Z \to G \times Z$ is the canonical embedding defined by $i(z) = (e, z)$ for $z \in Z$, and $\mathrm{pr} : G \times Z \to Z$, $q : G \times Z \to G$ are the canonical projections.

Let $L_Z : \mathfrak{g} \to \Theta_Z$ the canonical Lie algebra homomorphism explicitely given by

$$L_Z(g)(f)(z) = \frac{\partial}{\partial t} f(\exp(-tg)z)\Big|_{t=0} \tag{11}$$

for $g \in \mathfrak{g}$, $f \in \mathcal{O}_Z$, $z \in Z$. In particular, we denote by $L_G$ the canonical Lie algebra homomorphism defined by (11) with $Z = G$.

If $\mathcal{F}$ is a $G$-equivariant $\mathcal{O}_Z$-module, we obtain a Lie algebra homomorphism $\rho_{\mathcal{F}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{F})$ by setting

$$\rho_{\mathcal{F}}(g)(u) = i^* \left( (q^* L_G(g)) \right) \left( \beta_{\mathcal{F}}(\mu^* u) \right) \tag{12}$$

for $g \in \mathfrak{g}$ and $u \in \mathcal{F}$. It is a derivation since

$$\rho_{\mathcal{F}}(g)(fu) = (L_G(g)f)\, u + f\, (\rho_{\mathcal{F}}(g)(u))$$

for $g \in \mathfrak{g}$, $f \in \mathcal{O}_Z$ and $u \in \mathcal{F}$.

Assume now that $G$ acts transitively on $Z$. Fix $z \in Z$, denote by $H$ the the isotropy subgroup of $G$ at $z$ and identify $Z$ to $G/H$. If $\mathcal{F}$ is a $G$-equivariant $\mathcal{O}_Z$-module, its fiber $\mathbb{C} \otimes_{\mathcal{O}_{Z,z}} \mathcal{F}_z$ has a natural structure of $H$-module. Conversely, if $F$ is a $H$-module, we can define an $\mathcal{O}_Z$-module $\mathcal{O}_Z(F)$ by setting

$$\Gamma(U, \mathcal{O}_Z(F)) = \{ f \in \Gamma(\pi^{-1}(U), \mathcal{O}_G \otimes_{\mathbb{C}} F) \ : \ f(gh) = h^{-1} f(g) \quad \text{for any } g \in \pi^{-1}(U),\, h \in H \}$$

for $U$ open in $Z$, where $\pi : G \to G/H \cong Z$ is the canonical projection. The functor induced by the mapping $F \mapsto \mathcal{O}_Z(F)$ will be denoted by

$$\Psi_H : \mathrm{Mod}(H) \to \mathrm{Mod}_G(\mathcal{O}_Z).$$

Moreover, let us denote by $j_H : \{\mathrm{pt}\} \to G/H$ the map associated with $eH$. We have the following description of $G$-equivariant $\mathcal{O}_Z$-modules.

Proposition 2.1.17 (D. Mumford [**66**]). *If $Z = G/H$ where $H$ is a closed algebraic subgroup of $G$, the category $\mathrm{Mod}_G(\mathcal{O}_Z)$ is equivalent to the category $\mathrm{Mod}(H)$ consisting of $H$-modules via $\mathcal{F} \mapsto j_H^* \mathcal{F} = \mathbb{C} \otimes_{\mathcal{O}_{Z,eH}} \mathcal{F}_{eH}$ and $F \mapsto \Psi_H(F) = \mathcal{O}_Z(F)$.*

## 2.2. Quasi-equivariant $\mathcal{D}$-modules survey

Now, let us turn into the $\mathcal{D}$-module context, define the notion of $G$-equivariance for $\mathcal{D}$-modules and state an equivalence like the one given in proposition 2.1.17.

There are two options for equivariance in the $\mathcal{D}$-module case, one for $G$-equivariance and the other one, weaker, for the quasi-$G$-equivariance. Before giving the definition, let us denote by $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ the subring $\mathcal{O}_{G\times Z} \otimes_{\mathrm{pr}^{-1}\mathcal{O}_Z} \mathrm{pr}^{-1}\mathcal{D}_Z$ of $\mathcal{D}_{G\times Z}$.

Definition 2.2.1. A *quasi-$G$-equivariant $\mathcal{D}_Z$-module* is a $\mathcal{D}_Z$-module $\mathcal{M}$ endowed with a $\mathcal{O}_G \boxtimes \mathcal{D}_Z$-linear isomorphism

$$\beta_{\mathcal{M}} \ : \ \mathbf{D}\mu^*\mathcal{M} \to \mathbf{D}\,\mathrm{pr}^*\mathcal{M}$$

such that the following diagram commutes:

$$\begin{array}{ccccccc}
\mathbf{D}p_1^*\mathbf{D}\mu^*\mathcal{M} & & \xrightarrow{\quad\mathbf{D}p_1^*\beta_{\mathcal{M}}\quad} & & & & \mathbf{D}p_1^*\mathbf{D}\,\mathrm{pr}^*\mathcal{M} \\
\| & & & & & & \| \\
\mathbf{D}p_0^*\mathbf{D}\mu^*\mathcal{M} & \xrightarrow[\mathbf{D}p_0^*\beta_{\mathcal{M}}]{} & \mathbf{D}p_0^*\mathbf{D}\,\mathrm{pr}^*\mathcal{M} & \xrightarrow{\sim} & \mathbf{D}p_2^*\mathbf{D}\mu^*\mathcal{M} & \xrightarrow[\mathbf{D}p_2^*\beta_{\mathcal{M}}]{} & \mathbf{D}p_2^*\mathbf{D}\,\mathrm{pr}^*\mathcal{M}
\end{array}$$

where $\mathbf{D}p_1^*$, $\mathbf{D}\mu^*$ etc. are the pull-back functors for $\mathcal{D}$-modules. If $\beta$ is $\mathcal{D}_{G\times Z}$-linear, then $\mathcal{M}$ is called *$G$-equivariant*.

Examples 2.2.2. (i) If $\mathcal{F}$ is a $G$-equivariant $\mathcal{O}_Z$-module, then $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ is a quasi-$G$-equivariant $\mathcal{D}_Z$-module.

(ii) If $n \in \mathbb{N}_0$ and $P_1$, ..., $P_n$ are $G$-equivariant differential operators on $Z$, then $\mathcal{D}_Z/(\sum_{i=1}^n \mathcal{D}_Z \cdot P_i)$ is a quasi-$G$-equivariant $\mathcal{D}_Z$-module.

(iii) Suppose that $E$ is a $G$-equivariant algebraic vector bundle over $Z$ with a $G$-equivariant algebraic and flat connection $\nabla_E$. Denote by $\mathcal{O}(E)$ the sheaf of sections of $E$. Then $\mathcal{D}_Z$ acts on $\mathcal{O}(E)$ via the flat connection and $\mathcal{O}(E)$, seen as $\mathcal{D}_Z$-module, is quasi-$G$-equivariant.

Definition 2.2.3. For quasi-$G$-equivariant $\mathcal{D}_Z$-modules $\mathcal{M}$ and $\mathcal{N}$, a *$G$-equivariant morphism* $u : \mathcal{M} \to \mathcal{N}$ is a $\mathcal{D}_Z$-linear homomorphism $u : \mathcal{M} \to \mathcal{N}$ such that

$$\begin{array}{ccc}
\mathbf{D}\mu^*\mathcal{M} & \xrightarrow{\beta_{\mathcal{M}}} & \mathbf{D}\,\mathrm{pr}^*\mathcal{M} \\
{\scriptstyle \mathbf{D}\mu^* u}\downarrow & & \downarrow{\scriptstyle \mathbf{D}\,\mathrm{pr}^* u} \\
\mathbf{D}\mu^*\mathcal{N} & \xrightarrow[\beta_{\mathcal{N}}]{} & \mathbf{D}\,\mathrm{pr}^*\mathcal{N}
\end{array}$$

commutes.

DEFINITION 2.2.4. We denote by $\mathrm{Mod}(\mathcal{D}_Z, G)$ the category of quasi-coherent, quasi-$G$-equivariant $\mathcal{D}_Z$-modules and by $\mathrm{Mod}_G(\mathcal{D}_Z)$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_Z, G)$ consisting of $G$-equivariant $\mathcal{D}_Z$-modules.

Anticipating the sequel, the reader familiar with representation theory can remark that if $Z = \{\mathrm{pt}\}$, the category $\mathrm{Mod}(\mathcal{D}_Z, G)$ is the category $\mathrm{Mod}(G)$ consisting of $G$-modules.

Using the exactitude of the pullbacks functors $\mathrm{pr}^*$ and $\mu^*$, we obtain that $\mathrm{Mod}(\mathcal{D}_Z, G)$ and $\mathrm{Mod}_G(\mathcal{D}_Z)$ are abelian categories. Moreover, the forgetful functor $\mathrm{Mod}_G(\mathcal{D}_Z) \to \mathrm{Mod}(\mathcal{D}_Z, G)$ is fully faithful and exact. The forgetful functors

$$\mathrm{Mod}(\mathcal{D}_Z, G) \to \mathrm{Mod}(\mathcal{D}_Z) \to \mathrm{Mod}(\mathcal{O}_Z)$$

and

$$\mathrm{Mod}(\mathcal{D}_Z, G) \to \mathrm{Mod}_G(\mathcal{O}_Z)$$

are exact. We can see that the functor

$$\mathcal{D}_Z \otimes_{\mathcal{O}_Z} - \ : \ \mathrm{Mod}_G(\mathcal{O}_Z) \to \mathrm{Mod}(\mathcal{D}_Z, G)$$

is left adjoint to the forgetful functor $\mathrm{Mod}(\mathcal{D}_Z, G) \to \mathrm{Mod}_G(\mathcal{O}_Z)$. Explicitely, there is an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{F}, \mathcal{M}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}_Z, G)}(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}, \mathcal{M}) \tag{13}$$

which is functorial in $\mathcal{F} \in \mathrm{Mod}_G(\mathcal{O}_Z)$ and in $\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_Z, G)$.

**2.2.1. Derived categories of quasi-equivariant $\mathcal{D}$-modules.** Now we introduce the derived categories of quasi-$G$-equivariant $\mathcal{D}_Z$-modules.

COROLLARY 2.2.5. *If $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$, there exists a $G$-equivariant epimorphism $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F} \to \mathcal{M}$ where $\mathcal{F}$ is a locally free and coherent $G$-equivariant $\mathcal{O}_Z$-module.*

PROOF. Let $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$. Since $\mathcal{M}$ is $G$-equivariant as $\mathcal{O}_Z$-module, the proposition 2.1.12 gives a finite dimensional $G$-module $V$ and a $G$-equivariant epimorphism

$$\alpha \ : \ V \otimes \mathcal{L}^{\otimes -n} \to \mathcal{M}$$

in $\mathrm{Mod}_G(\mathcal{O}_Z)$. Take $\mathcal{F} = V \otimes \mathcal{L}^{\otimes -n}$. This is a $G$-equivariant $\mathcal{O}_Z$-module which is locally free of rank $n$. The conclusion follows by using the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(\mathcal{F}, \mathcal{M}) \cong \mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}_Z, G)}(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}, \mathcal{M}).$$

□

Thus we see that the category where objects are $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ with such $\mathcal{F}$ plays an important role in $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$ since they are in some sense the analogues of the locally free $\mathcal{D}_Z$-modules of the non-equivariant setting.

Definition 2.2.6. We denote by $\mathrm{Mod}_{\mathrm{lf}}(\mathcal{D}_Z, G)$ the full subcategory of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$ where objects are of the form $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}$ for a locally free coherent $G$-equivariant $O_Z$-module $\mathcal{F}$.

We established in the corollary 2.2.5 that the category $\mathrm{Mod}_{\mathrm{lf}}(\mathcal{D}_Z, G)$ is generating in $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$. Moreover, one can show that every object of $\mathrm{Mod}(\mathcal{D}_Z, G)$ has a finite resolution by objects of $\mathrm{Mod}(\mathcal{D}_Z, G)$ which are flat over $\mathcal{O}_Z$. In fact, the category $\mathrm{Mod}(\mathcal{D}_Z, G)$ is a Grothendieck category and as consequence, some standard arguments of homological algebra allows to prove the following proposition:

Proposition 2.2.7. *Any object of* $\mathrm{Mod}(\mathcal{D}_Z, G)$ *is embedded in an injective object of* $\mathrm{Mod}(\mathcal{D}_Z, G)$*.*

Proof. See [**51**] p.236. □

Definition 2.2.8. We denote by $\mathbf{D}^b(\mathcal{D}_Z, G)$ and $\mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Z, G)$ the derived categories built from $\mathrm{Mod}(\mathcal{D}_Z, G)$ and $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$ respectively.

The forgetful functor $\mathrm{Mod}(\mathcal{D}_Z, G) \to \mathrm{Mod}_G(\mathcal{O}_Z)$ sends injectives to injectives, as states the following proposition:

Proposition 2.2.9. *Denote by* $\mathcal{F} : \mathrm{Mod}(\mathcal{D}_Z, G) \to \mathrm{Mod}_G(\mathcal{O}_Z)$ *the forgetful functor. If* $\mathcal{I}$ *is an injective object of* $\mathrm{Mod}(\mathcal{D}_Z, G)$*, then* $\mathcal{F}\mathcal{I}$ *is also injective in* $\mathrm{Mod}_G(\mathcal{O}_Z)$*.*

Proof. Since $\mathcal{I}$ is injective, the functor $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}_Z, G)}(-, \mathcal{I})$ is exact and its composition with the exact functor which maps $\mathcal{G} \in \mathrm{Mod}_G(\mathcal{O}_Z)$ to $\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{G} \in \mathrm{Mod}(\mathcal{D}_Z, G)$ is still exact. Thus the isomorphism (13) then gives that the functor $\mathrm{Hom}_{\mathrm{Mod}_G(\mathcal{O}_Z)}(-, \mathcal{F}\mathcal{I})$ is exact too. □

Finally, now that the derived category of quasi-equivariant $\mathcal{D}$-modules is defined, we conclude this section with the following result, which will be useful for interpreting the direct image functor of $\mathcal{D}$-modules in the quasi-equivariant setting.

Proposition 2.2.10. *Suppose that $X$ and $Y$ are two quasi-projective algebraic $G$-manifolds. Let $f : X \to Y$ be a $G$-equivariant morphism. If $\mathcal{I}$ is an injective object of* $\mathrm{Mod}(\mathcal{D}_X, G)$ *and $\mathcal{E}$ is a locally free $G$-equivariant $\mathcal{O}_X$-module, then*

$$R^k f_*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}) = 0$$

*for every $k \in \mathbb{N}_0$.*

Proof. This is a consequence of proposition 2.1.16. □

### 2.2.2. Operations on quasi-equivariant $\mathcal{D}$-modules.

2.2.2.1. *Internal and external tensor products.* Similarly to the case of non-equivariant $\mathcal{D}$-modules, we define the internal and external tensor products for quasi-$G$-equivariant $\mathcal{D}$-modules.

Let $\mathcal{M}$, $\mathcal{N}$ be two quasi-$G$-equivariant $\mathcal{D}_Z$-modules. The $\mathcal{O}_Z$-module $\mathcal{M} \otimes_{\mathcal{O}_Z} \mathcal{N}$ has a structure of $\mathcal{D}_Z$-module induced by

$$\theta(m \otimes n) = (\theta m) \otimes n + m \otimes (\theta n)$$

where $\theta \in \Theta_Z$, $m \in \mathcal{M}$, $n \in \mathcal{N}$. The $\mathcal{D}_Z$-module $\mathcal{M} \otimes_{\mathcal{O}_Z} \mathcal{N}$ is denoted by $\mathcal{M} \otimes^D \mathcal{N}$. Since the $\mathcal{D}_Z$-module structure on $\mathcal{M} \otimes^D \mathcal{N}$ is $G$-equivariant, we obtain a right exact bi-functor

$$- \overset{D}{\otimes} - \ : \ \mathrm{Mod}(\mathcal{D}_Z,\, G) \times \mathrm{Mod}(\mathcal{D}_Z,\, G) \to \mathrm{Mod}(\mathcal{D}_Z,\, G).$$

Taking its left derived functor, we obtain

$$- \overset{D}{\otimes} - \ : \ \mathbf{D}^b(\mathcal{D}_Z,\, G) \times \mathbf{D}^b(\mathcal{D}_Z,\, G) \to \mathbf{D}^b(\mathcal{D}_Z,\, G).$$

LEMMA 2.2.11. *Let $\mathcal{F}$ be a $G$-equivariant $\mathcal{O}_Z$-module and $\mathcal{M}$ be a quasi-$G$-equivariant $\mathcal{D}_Z$-module. Regarding $\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{M}$ as a $G$-equivariant $\mathcal{O}_Z$-module, there is a canonical isomorphism*

$$(\mathcal{D}_Z \otimes_{\mathcal{O}_Z} \mathcal{F}) \otimes \mathcal{M} \cong \mathcal{D}_Z \otimes_{\mathcal{O}_Z} (\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{M})$$

*as quasi-$G$-equivariant $\mathcal{D}_Z$-modules.*

PROOF. The proof is similar to the one related to the non-equivariant setting. See **[48]** for instance. □

Moreover, if $X$ and $Y$ are two algebraic $G$-manifolds and $p_X$ (resp. $p_Y$) denotes the canonical projection of $X \times Y$ on $X$ (resp. $Y$), we can define a quasi-$G$-equivariant $\mathcal{D}_{X\times Y}$-module $\mathcal{M} \boxtimes \mathcal{N}$ by

$$\mathcal{M} \boxtimes \mathcal{N} = \left(\mathcal{O}_{X\times Y} \otimes_{p_X^{-1}\mathcal{O}_X} p_X^{-1}\mathcal{M}\right) \otimes_{p_Y^{-1}\mathcal{O}_Y} p_Y^{-1}\mathcal{N}.$$

This object of $\mathrm{Mod}(\mathcal{D}_{X\times Y},\, G)$ is called the *exterior tensor product of $\mathcal{M}$ by $\mathcal{N}$* and since the mapping $(\mathcal{M}, \mathcal{N}) \mapsto \mathcal{M} \boxtimes \mathcal{N}$ obviously defines an exact bifunctor, we get a derived functor:

$$- \boxtimes - \ : \ \mathbf{D}^b(\mathcal{D}_X,\, G) \times \mathbf{D}^b(\mathcal{D}_Y,\, G) \to \mathbf{D}^b(\mathcal{D}_{X\times Y},\, G).$$

Taking $Y = \{\mathrm{pt}\}$, we obtain a functor

$$- \otimes - \ : \ \mathbf{D}^b(\mathcal{D}_X,\, G) \times \mathbf{D}^b(G) \to \mathbf{D}^b(\mathcal{D}_X,\, G).$$

2.2.2.2. *Semi-outer Hom bifunctor.* If $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$ and $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Z, G)$, then the complex vector space $\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$ has a structure of $G$-module. To describe this structure, we first note that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N}) &\to \mathrm{Hom}_{\mathcal{D}_{G\times Z}}(\mathbf{D}\mu^*\mathcal{M}, \mathbf{D}\mu^*\mathcal{N}) \\ &\to \mathrm{Hom}_{\mathcal{O}_G\boxtimes\mathcal{D}_Z}(\mathbf{D}\mu^*\mathcal{M}, \mathbf{D}\mu^*\mathcal{N}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_G\boxtimes\mathcal{D}_Z}(\mathbf{D}\,\mathrm{pr}^*\mathcal{M}, \mathbf{D}\,\mathrm{pr}^*\mathcal{N}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathrm{pr}_*\mathbf{D}\,\mathrm{pr}^*\mathcal{N}).\end{aligned}$$

Since $\mathcal{N}$ is quasi-coherent, projection formulas gives

$$\begin{aligned}\mathrm{pr}_*\mathbf{D}\,\mathrm{pr}^*\mathcal{N} &\cong \mathrm{pr}_*\mathcal{D}_{G\times Z\to Z}\otimes_{\mathcal{D}_Z}\mathcal{N} \cong (\mathrm{pr}_*\mathcal{O}_{G\times Z}\otimes_{\mathcal{O}_Z}\mathcal{D}_Z)\otimes_{\mathcal{D}_Z}\mathcal{N} \\ &\cong \Gamma(G, \mathcal{O}_G)\otimes\mathcal{N}\end{aligned}$$

and we obtain finally a morphism

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N}) &\to \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \Gamma(G, \mathcal{O}_G)\otimes\mathcal{N}) \\ &\cong \Gamma(G, \mathcal{O}_G)\otimes\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})\end{aligned}$$

using the coherency of $\mathcal{M}$. If $\xi \in \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$, let us denote by $f\otimes\xi'$ the image of $\xi$ by this morphism, where $f \in \Gamma(G, \mathcal{O}_G)$ and $\xi' \in \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$. Then the action of $G$ on $\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$ is given by

$$g\cdot\xi = f(g)\xi'$$

for every $g \in G$, $\xi \in \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$.

Hence we obtain a bifunctor

$$\mathrm{Hom}_{\mathcal{D}_Z}(-, -) : \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)^{\mathrm{op}}\times\mathrm{Mod}(\mathcal{D}_Z, G)\to\mathrm{Mod}(G).$$

Similarly to the classical case, when $\mathcal{M}$ is coherent, the functor $\mathcal{M}\otimes -$ is a left adjoint to $\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, -)$ in the equivariant context:

PROPOSITION 2.2.12. *We have*

$$\mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}_Z, G)}(\mathcal{M}\otimes M, \mathcal{N}) \cong \mathrm{Hom}_G(M, \mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N}))$$

*for every* $M \in \mathrm{Mod}(G)$, $\mathcal{M} \in \mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$ *and* $\mathcal{N} \in \mathrm{Mod}(\mathcal{D}_Z, G)$.

COROLLARY 2.2.13. *If* $\mathcal{I}$ *is an injective object of* $\mathrm{Mod}(\mathcal{D}_Z, G)$ *and* $\mathcal{M}$ *is an object of* $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$*, then* $\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{I})$ *is an injective object of* $\mathrm{Mod}(G)$.

PROOF. This follows immediately from the exactness of the functor

$$\mathrm{Mod}(G)\to\mathrm{Mod}(\mathcal{D}_Z, G) : V\to V\otimes\mathcal{M}$$

and from the proposition 2.2.12. □

Definition 2.2.14. For every $\mathcal{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Z, G)$, we define

$$\mathrm{RHom}_{\mathcal{D}_Z}(\mathcal{M}, -) \ : \ \mathbf{D}^+(\mathcal{D}_Z, G) \to \mathbf{D}^+(G)$$

to be the right derived functor of $\mathrm{Hom}_{\mathcal{D}_Z}(\mathcal{M}, -)$. We get the functor

$$\mathrm{RHom}_{\mathcal{D}_Z}(-, -) \ : \ \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Z, G)^{\mathrm{op}} \times \mathbf{D}^+(\mathcal{D}_Z, G) \to \mathbf{D}^+(G).$$

Proposition 2.2.15. *If $\mathcal{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Z, G)$ and $\mathcal{N} \in \mathbf{D}^b(\mathcal{D}_Z, G)$, then $\mathrm{RHom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})$ belongs to $\mathbf{D}^b(G)$.*

Proof. This follows from the fact that the category $\mathrm{Mod}(\mathcal{D}_Z)$ has finite global homological dimension. See **[49]** for details. □

Getting into derived categories, we obtain the following analogue to proposition 2.2.12:

Proposition 2.2.16. *Suppose that $\mathcal{M}$ and $\mathcal{N}$ are objects of $\mathbf{D}^b(\mathcal{D}_Z, G)$. If $\mathcal{M}$ is coherent and $M \in \mathbf{D}^b(G)$, then*

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_Z, G)}(\mathcal{M} \otimes M, \mathcal{N}) \cong \mathrm{Hom}_{\mathbf{D}^b(G)}(M, \mathrm{RHom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})).$$

*In particular, we have*

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_Z, G)}(\mathcal{M}, \mathcal{N}) \cong \mathrm{Hom}_{\mathbf{D}^b(G)}(\mathbb{C}, \mathrm{RHom}_{\mathcal{D}_Z}(\mathcal{M}, \mathcal{N})).$$

The latter isomorphism will be a crucial tool in our further developments (see proposition 2.2.17).

**2.2.3. Inverse and direct image functors and equivariant adjunction.** In this subsection, we define the inverse image and direct image functors for $\mathcal{D}$-modules, in the $G$-equivariant setting. We conclude by an equivariant analogue of the classical adjunction theorem between those two fundamental $\mathcal{D}$-modules operations.

Let $X$ (resp. $Y$) be a quasi-projective algebraic $G$-manifold of complex dimension $d_X$ (resp. $d_Y$) and set $d_{Y/X} = d_Y - d_X$. Let $f \ : X \to Y$ be a $G$-equivariant morphism. Exactly as in the non-equivariant case, denote by

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y$$

the transfer $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule used for the $\mathcal{D}$-module inverse image functor and by

$$\mathcal{D}_{Y\leftarrow X} = f^{-1}\mathcal{D}_Y \otimes_{f^{-1}\mathcal{O}_Y} \Omega_{X/Y}$$

the $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodule for the $\mathcal{D}$-module direct image, where

$$\Omega_{X/Y} = \Omega_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\Omega_Y^{\otimes -1}.$$

The functor

$$f^* \ : \ \mathrm{Mod}(\mathcal{D}_Y, G) \to \mathrm{Mod}(\mathcal{D}_X, G) \ : \ \mathcal{N} \mapsto \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{N}$$

defines a right exact functor which can be left derived in the bounded derived category since, as we seen in subsection 2.2.1, any quasi-coherent quasi-$G$-equivariant $\mathcal{D}_Y$-module has a finite resolution by quasi-coherent quasi-$G$-equivariant $\mathcal{D}_Y$-modules which are flat over $\mathcal{O}_Y$. Thus we get a derived inverse image functor

$$\mathbf{D}f^* \ : \ \mathbf{D}^b(\mathcal{D}_Y, G) \to \mathbf{D}^b(\mathcal{D}_X, G).$$

The usual definition of direct image functor in the non-equivariant setting

$$\mathcal{M} \in \mathrm{Mod}(\mathcal{D}_X) \mapsto Rf_*(\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}) \in \mathrm{Mod}(\mathcal{D}_Y)$$

can also be interpreted in bounded derived category in the quasi-equivariant setting by taking a de Rham-type resolution for $\mathcal{D}_{Y\leftarrow X}$ with $(f^{-1}\mathcal{D}_Y, \mathcal{D}_X)$-bimodules which are flat over $\mathcal{D}_X$ and representing $\mathcal{M}$ with an injective complex. Explicitely, denote by $d_X$ (resp. $d_Y$) the complex dimension of $X$ (resp. $Y$) and take the following resolution of $\mathcal{D}_{Y\leftarrow X}$ by flat $\mathcal{D}_X$-modules:

$$\begin{aligned} 0 \leftarrow \mathcal{D}_{Y\leftarrow X} \ &\leftarrow \ f^{-1}\left(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\right) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ &\leftarrow \ f^{-1}\left(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\right) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{d_X-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ &\leftarrow \ \dots \\ &\leftarrow \ f^{-1}\left(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\right) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{0} \otimes_{\mathcal{O}_X} \mathcal{D}_X \leftarrow 0. \end{aligned}$$

Thus, if $\mathcal{M}$ is a complex of $\mathcal{D}_X$-modules, the object $\mathcal{D}_{Y\leftarrow X} \otimes^L_{\mathcal{D}_X} \mathcal{M}$ is represented by the complex of $f^{-1}\mathcal{D}_Y$-modules

$$f^{-1}\left(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\right) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}[d_X].$$

We define the following functor on the level of left-bounded homotopy categories

$$\mathrm{K}f_* \ : \ \mathrm{K}^+\left(\mathrm{Mod}(\mathcal{D}_X, G)\right) \to \mathrm{K}^+\left(\mathrm{Mod}(\mathcal{D}_Y, G)\right)$$

by

$$\begin{aligned} \mathrm{K}f_*(\mathcal{M}) \ &= \ f_*\left(f^{-1}\left(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1}\right) \otimes_{f^{-1}\mathcal{O}_Y} \Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}[d_X]\right) \\ &\cong \ \mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1} \otimes_{\mathcal{O}_Y} f_*\left(\Omega_X^{\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}\right)[d_X] \end{aligned}$$

Using the proposition 2.2.10, we obtain that if $\mathcal{I}^\cdot$ is an exact complex in $\mathrm{Mod}(\mathcal{D}_X, G)$ with injective components, then the complex $\mathrm{K}f_*\mathcal{I}^\cdot$ is also exact in $\mathrm{Mod}(\mathcal{D}_Y, G)$. Hence $\mathrm{K}f_*$ is right derivable and we can denote by $\mathbf{D}f_*$ its right derived functor:

$$\mathbf{D}f_* \ : \ \mathbf{D}^+(\mathcal{D}_X, G) \to \mathbf{D}^+(\mathcal{D}_Y, G).$$

The following proposition shows that in the quasi-$G$-equivariant case, the equivariant inverse image functor $\mathbf{D}f^*[d_{Y/X}]$ is a left adjoint to the equivariant direct image functor $\mathbf{D}f_*$ under the classical coherency hypothesis.

PROPOSITION 2.2.17. *If $f : X \to Y$ is a smooth morphism between two quasi-projective algebraic $G$-manifolds, then*

$$\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_X,\,G)}(\mathbf{D}f^*\mathcal{M}[d_{Y/X}],\,\mathcal{N}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_Y,\,G)}(\mathcal{M},\,\mathbf{D}f_*\mathcal{N})$$

*for every $\mathcal{M} \in \mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Y,\,G)$ and $\mathcal{N} \in \mathbf{D}^b(\mathcal{D}_X,\,G)$,*

PROOF. Let us abbreviate $d_{Y/X}$ by $d$. On one hand, forgetting the $G$-equivariance, it is known (see [**48**]) that the canonical morphism

$$\varphi_{\mathcal{M}} \;:\; \mathcal{M} \to \mathbf{D}f_*\mathbf{D}f^*\mathcal{M}[d_{Y/X}]$$

induces an isomorphism

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathbf{D}f^*\mathcal{M}[d_{Y/X}],\,\mathcal{N}) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M},\,\mathbf{D}f_*\mathcal{N}) \tag{14}$$

in $\mathbf{D}^b(\mathbb{C})$. On the other hand, the coherence of $\mathcal{M}$ and the $G$-equivariance give that the classical isomorphism

$$\mathrm{RHom}_{\mathcal{D}_X}(\mathbf{D}f^*\mathcal{M}[d],\,\mathcal{N}) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M},\,\mathbf{D}f_*\mathcal{N}) \tag{15}$$

is an isomorphism in $\mathbf{D}^b(G)$. Hence, the proposition 2.2.16 gives that

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_X,\,G)}(\mathbf{D}f^*\mathcal{M}[d],\,\mathcal{N}) &\cong \mathrm{Hom}_{\mathbf{D}^b(G)}(\mathbb{C},\,\mathrm{RHom}_{\mathcal{D}_X}(\mathbf{D}f^*\mathcal{M}[d],\,\mathcal{N}))\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{D}^b(G)}(\mathbb{C},\,\mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M},\,\mathbf{D}f_*\mathcal{N}))\\
&\cong \mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_Y,\,G)}(\mathcal{M},\,\mathbf{D}f_*\mathcal{N})
\end{aligned}$$

and the proof is complete. □

**2.2.4. Quasi-equivariant $\mathcal{D}$-modules and representation theory.** In the section 2.1.2, we obtained a correspondence between equivariant $\mathcal{O}$-modules and representations of Lie groups. The aim of this subsection is to explain a similar equivalence in the $\mathcal{D}$-module case, which was established by M. KASHIWARA (see [**47**]).

Before getting into the general case, we would like to motivate and illustrate the quest of the $\mathcal{D}$-module equivalence from the third example of quasi-$G$-equivariant $\mathcal{D}_Z$-module previously given (see p.48). If $E$ is a $G$-equivariant algebraic vector bundle over $Z$ with a $G$-equivariant algebraic flat connection $\nabla_E$, the sheaf $\mathcal{O}(E)$ of sections of $E$ is a quasi-$G$-equivariant $\mathcal{D}_Z$-module. If $G$ acts transitively on $Z$, the equivariant vector bundles over $Z$ with flat equivariant connections are the only examples of quasi-$G$-equivariant $\mathcal{D}_Z$-modules which are coherent over $\mathcal{O}_Z$. In fact, there is an equivalence of categories between the category of quasi-$G$-equivariant and $\mathcal{O}_Z$-coherent $\mathcal{D}_Z$-modules and the category of pairs $(E,\,\nabla_E)$ consisting of a $G$-equivariant algebraic vector bundle $E$ over $Z$ with a $G$-equivariant algebraic flat connection $\nabla_E$

(in this category, the maps are the flat $G$-equivariant bundle maps between such objects). In turn, the latter category can be identified with the category of finite dimensional algebraic $(\mathfrak{g}, H)$-modules (see definition 2.2.19) where $H$ denotes the isotropy subgroup of $G$ at some reference point $z \in Z$. The aim of this section is to show that this equivalence between quasi-equivariant $\mathcal{D}$-modules and $(\mathfrak{g}, H)$-modules remains valid even without the coherency hypothesis, but allowing the algebraic vectors bundles to be of infinite rank and allowing the algebraic $(\mathfrak{g}, H)$-modules to be infinite dimensional.

Let $\mathcal{M}$ be a quasi-$G$-equivariant $\mathcal{D}_Z$-module. The $G$-equivariant $\mathcal{O}_Z$-module structure on $\mathcal{M}$ induces a Lie algebra homomorphism $\rho_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$ (see section 2.1.2). Moreover, the $\mathcal{D}_Z$-module structure on $\mathcal{M}$ induces the Lie algebra homomorphism

$$\kappa_{\mathcal{M}} : \mathfrak{g} \to \Gamma(Z, \mathcal{D}_Z) \to \mathrm{End}_{\mathbb{C}}(\mathcal{M})$$

defined by

$$\kappa_{\mathcal{M}}(g)(m) = L_Z(g)(m)$$

for $g \in \mathfrak{g}$, $m \in \mathcal{M}$. Define $\gamma_{\mathcal{M}} = \rho_{\mathcal{M}} - \kappa_{\mathcal{M}}$.

The main result is the following:

Proposition 2.2.18 (M. Kashiwara [**47**] pp.85–86). *Let $\mathcal{M}$ be a quasi-$G$-equivariant $\mathcal{D}_Z$-module. Then*

(1) *$\gamma_{\mathcal{M}}(g)$ belongs to $\mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ for each $g \in \mathfrak{g}$;*
(2) *the map $\gamma_{\mathcal{M}} : \mathfrak{g} \to \mathrm{End}_{\mathcal{D}_Z}(\mathcal{M})$ is a Lie algebra homomorphism;*
(3) *$\mathcal{M}$ is $G$-equivariant if and only if $\gamma_{\mathcal{M}} = 0$.*

Now, exactly like the $\mathcal{O}$-module case, suppose that $G$ acts transitively on $Z$ and write $Z = G/H$ where $H$ is a closed algebraic subgroup of $G$. If $\mathcal{M}$ is a quasi-$G$-equivariant $\mathcal{D}_Z$-module, then its fiber $j_H^*\mathcal{M}$ is endowed with an action of $H$ coming from the $\mathcal{O}_Z$-module structure of $\mathcal{M}$. Moreover, $j_H^*\mathcal{M}$ is also endowed with a $\mathfrak{g}$-module structure induced from the $\mathcal{O}_Z$-linear action $\gamma_{\mathcal{M}}$. If we denote by $M = j_H^*\mathcal{M}$ and by $\mathfrak{h}$ the Lie subalgebra of $\mathfrak{g}$ associated to $H$, we can check the following (see [**47**]):

(1) the $\mathfrak{h}$-module structure on $M$ induced by differentiation by the $H$-module structure coincides with the restriction of the $\mathfrak{g}$-module structure;
(2) the multiplication homomorphism $\mathfrak{g} \otimes M \to M$ is $H$-linear, where $H$ acts on $\mathfrak{g}$ by the adjoint action, i.e.

$$h(gm) = (\mathrm{Ad}(h)g)(gm)$$

for all $h \in H$, $g \in \mathfrak{g}$ and $m \in M$.

Definition 2.2.19. A vector space equipped with structures of $H$- and $\mathfrak{g}$-module is called a $(\mathfrak{g}, H)$*-module* if it meets the conditions (1) and (2) above. A morphism of $(\mathfrak{g}, H)$-modules is a linear map which is a morphism for both $\mathfrak{g}$- and $H$-module structures. We denote by $\mathrm{Hom}_{(\mathfrak{g}, H)}(M, N)$ the complex vector space of all morphisms between the $(\mathfrak{g}, H)$-modules $M$ and $N$, and by $\mathrm{Mod}(\mathfrak{g}, H)$ the category of $(\mathfrak{g}, H)$-modules. Denote by $\mathbf{D}^b(\mathfrak{g}, H)$ the derived category built from $\mathrm{Mod}(\mathfrak{g}, H)$.

EXAMPLE 2.2.20. Let $Q_I$ a standard parabolic subgroup associated to a subset $I$ of $I_0$. Take $\lambda$ integer and dominant for $Q_I$ and denote by $M_I(\lambda)$ the generalized Verma module associated to $\lambda$ (defined p18). It is a fundamental example of $(\mathfrak{g}, Q_I)$-module with actions defined by

$$g \cdot (g' \otimes v) = (gg') \otimes v, \quad g \in \mathfrak{g},\ g' \otimes v \in M_I(\lambda)$$

and

$$X \cdot (g' \otimes v) = g' \otimes (Xv), \quad X \in Q_I,\ g' \otimes v \in M_I(\lambda).$$

Therefore the functor $\mathcal{M} \mapsto j_H^*\mathcal{M} = \mathbb{C} \otimes_{\mathcal{O}_{Z,eH}} \mathcal{M}_{eH}$ sends $\mathrm{Mod}(\mathcal{D}_Z, G)$ to $\mathrm{Mod}(\mathfrak{g}, H)$. Conversely, if $M$ is a $(\mathfrak{g}, H)$-module, there is a Lie algebra homomorphism $\gamma_{\mathcal{O}_Z(M)} : \mathfrak{g} \to \mathrm{End}_{\mathcal{O}_Z}(\mathcal{O}_Z(M))$ and $\rho_{\mathcal{O}_Z(M)} - \gamma_{\mathcal{O}_Z(M)}$ defines a $\mathcal{D}_Z$-module structure on $\mathcal{O}_Z(M)$. The main result of this section is the following correspondence of categories:

PROPOSITION 2.2.21 (M. KASHIWARA [**47**] PP.85–86). *If $Z = G/H$ where $H$ is a closed algebraic subgroup of $G$, then the category $\mathbf{D}^b(\mathcal{D}_Z, G)$ is equivalent to the category $\mathbf{D}^b(\mathfrak{g}, H)$ via the correspondences $\mathcal{M} \mapsto j_H^*\mathcal{M}$ and $M \mapsto \Psi_H(M) = \mathcal{O}_Z(M)$.*

CHAPTER 3

# Algebraic analogs of image functors in quasi-equivariant $\mathcal{D}$-modules categories

This chapter contains the results obtained in this thesis, presented as pedagogically as possible. We deal here with quasi-equivariant algebraic $\mathcal{D}$-modules and M. KASHIWARA's correspondence 2.2.21 is used to determine the analogues of the $\mathcal{D}$-module inverse and direct image functors in the algebraic side where objects are representation spaces of Lie groups and Lie algebras. In particular, we will show that the analog of the inverse image functor for $\mathcal{D}$-modules is the forgetful functor of some of the group structure, and that the analog of the direct image functor for $\mathcal{D}$-modules is the Zuckerman functor. We use the Lie algebra and Lie group tools contained of chapter 1 in a smooth complex algebraic context, since all useful results can be obtained in that algebraic setting.

The results previously obtained are then used in a framework where algebraic varieties are generalized flag manifolds. In section 3.3, we will compute the image of generalized Verma modules with given highest weight by the Zuckerman derived functor. This proves in theorem 3.3.2 an algebraic version of the classical Bott-Borel-Weil theorem.

Finally, in section 3.4, we describe the algebraic variant of the generalized Penrose transform studied in the quasi-equivariant setting by C. MARASTONI and T. TANISAKI in [**64**]. The Radon-Penrose transformation of quasi-equivariant $\mathcal{D}$-modules is transported to the algebraic side and we describe a way to compute the algebraic variant of the Radon-Penrose transformation.

## 3.1. Inverse image functor and forgetful functor

When $G$ acts transitively on $X$ and $Y$ and $f$ is a projection, the algebraic analog of the inverse image functor which is induced by the equivalence of categories 2.2.18 is a forgetful functor. To see it, take two closed algebraic subgroups $H \subset K$ of $G$ and denote by $g : G/H \to G/K$ the canonical projection. For $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_{G/K}, G)$, the quasi-$G$-equivariance of $\mathcal{M}$ gives that

$$\mathbb{C} \otimes_{\mathcal{O}_{G/H,\,eH}} (\mathbf{D}g^*\mathcal{M})_{eH} \cong \mathbb{C} \otimes_{\mathcal{O}_{G/K,\,eK}} \mathcal{M}_{eK}$$

as $(\mathfrak{g}, H)$-modules. Thus we have obtained the following result:

PROPOSITION 3.1.1. *The diagram*

$$\begin{array}{ccc} \mathbf{D}^b(\mathcal{D}_{G/K},\, G) & \xrightarrow{\mathbf{D}g^*} & \mathbf{D}^b(\mathcal{D}_{G/H},\, G) \\ {\scriptstyle j_K^* \wr}\downarrow & & \downarrow{\scriptstyle \wr\, j_H^*} \\ \mathbf{D}^b(\mathfrak{g},\, K) & \xrightarrow[\mathcal{F}_K^H]{} & \mathbf{D}^b(\mathfrak{g},\, H) \end{array} \tag{16}$$

*is commutative, where* $\mathcal{F}_K^H\ :\ \mathrm{Mod}(\mathfrak{g},\, K) \to \mathrm{Mod}(\mathfrak{g},\, H)$ *is the functor which forgets some of the group structure.*

Define $\gamma_K^H$ to be the isomorphism of functors $\mathbf{D}g^*\Psi_K \xrightarrow{\sim} \Psi_H\mathcal{F}_K^H$.

### 3.2. Direct image functor and Zuckerman functor

Now let $H \subset K$ be two closed algebraic subgroups of $G$. The right adjoint to the forgetful functor $\mathcal{F}_K^H\ : \mathrm{Mod}(\mathfrak{g},\, K) \to \mathrm{Mod}(\mathfrak{g},\, H)$ is called *Zuckerman functor* and is denoted by $\Gamma_H^K$.

Set $\delta$ the canonical morphism $\mathcal{F}_K^H \mathrm{R}\Gamma_H^K \to \mathrm{id}$.

When $N \in \mathbf{D}^b(\mathfrak{g},\, K)$ and $N' \in \mathbf{D}^b(\mathfrak{g},\, H)$, denote by $\xi'$ the image in $\mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\, H)}(\mathcal{F}_K^H N,\, N')$ of $\xi \in \mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\, K)}(N,\, \mathrm{R}\Gamma_H^K N')$, i.e.

$$\xi' = \delta(N')\mathcal{F}_K^H(\xi).$$

Denote by $g\ :\ G/H \to G/K$ the canonical projection and by

$$\beta\ :\ \mathrm{id}[-(d_K - d_H)] \to \mathbf{D}g_*\mathbf{D}g^*$$

the canonical morphism.

THEOREM 3.2.1. *There is an isomorphism*

$$\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-(d_K - d_H)] \to j_K^*\mathbf{D}g_*\mathcal{M}$$

*when* $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_{G/H},\, G)$.

PROOF. Fix $\mathcal{M} \in \mathbf{D}^b(\mathcal{D}_{G/H},\, G)$ and $N \in \mathbf{D}^b(\mathfrak{g},\, K)$. In this proof, let us abbreviate $d_K - d_H$ by $d$ for sake of clarity. The composition

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\, K)}(N,\, \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]) \\ \xrightarrow{\sim}\quad &\mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\, H)}(\mathcal{F}_K^H N,\, j_H^*\mathcal{M}[-d]) \\ \xrightarrow{\sim}\quad &\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_{G/H},\, G)}(\Psi_H\mathcal{F}_K^H N,\, \Psi_H j_H^*\mathcal{M}[-d]) \\ \xrightarrow{\sim}\quad &\mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_{G/H},\, G)}(\Psi_H\mathcal{F}_K^H N,\, \mathcal{M}[-d]) \end{aligned}$$

$$
\begin{aligned}
&\xrightarrow{\sim} \quad \mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_{G/H},\,G)}(\mathbf{D}g^*\Psi_K N,\, \mathcal{M}[-d]) \\
&\xrightarrow{(*)} \quad \mathrm{Hom}_{\mathbf{D}^b(\mathcal{D}_{G/K},\,G)}(\Psi_K N,\, \mathbf{D}g_*\mathcal{M}) \\
&\xrightarrow{\sim} \quad \mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\,K)}(j_K^*\Psi_K N,\, j_K^*\mathbf{D}g_*\mathcal{M}) \\
&\xrightarrow{\sim} \quad \mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\,K)}(N,\, j_K^*\mathbf{D}g_*\mathcal{M})
\end{aligned}
$$

is denoted by $G$ and sends $\xi \in \mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\,K)}(N,\, \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d])$ to

$$
\begin{aligned}
G(\xi) = {}& [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ \left[j_K^*\mathbf{D}g_*\left(\Psi_H(\xi')[d]\right)\right] \circ \\
&\qquad \left[j_K^*\mathbf{D}g_*\gamma_K^H(N)[d]\right] \circ [j_K^*\beta(\Psi_K(N)[d])] \circ \alpha_K(N).
\end{aligned}
$$

Since $\xi' = \delta\left(j_H^*\mathcal{M}[-d]\right) \circ \mathcal{F}_K^H(\xi)$, we have

$$
\begin{aligned}
G(\xi) = {}& [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ [j_K^*\mathbf{D}g_*\Psi_H\left(\delta\left(j_H^*\mathcal{M}\right)\right)] \circ \\
&\left[j_K^*\mathbf{D}g_*\Psi_H\mathcal{F}_K^H(\xi)[d]\right] \circ \left[j_K^*\mathbf{D}g_*\gamma_K^H(N)[d]\right] \circ [j_K^*\beta(\Psi_K(N)[d])] \circ \alpha_K(N).
\end{aligned}
$$

Moreover, since $\gamma_K^H$ is a morphism of functors, the diagram

$$
\begin{array}{ccc}
\mathbf{D}g^*\Psi_K(N) & \xrightarrow{\quad \gamma_K^H(N) \quad} & \Psi_H\mathcal{F}_K^H(N) \\
\Big\downarrow{\scriptstyle \mathbf{D}g^*\Psi_K(\xi)} & & \Big\downarrow{\scriptstyle \Psi_H\mathcal{F}_K^H(\xi)} \\
\mathbf{D}g^*\Psi_K\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right) & \xrightarrow[\gamma_K^H\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right)]{} & \Psi_H\mathcal{F}_H^K\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right)
\end{array}
$$

is commutative and we get

$$
\begin{aligned}
G(\xi) = {}& [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ [j_K^*\mathbf{D}g_*\Psi_H\left(\delta\left(j_H^*\mathcal{M}\right)\right)] \circ \\
&\left[j_K^*\mathbf{D}g_*\gamma_K^H\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ [j_K^*\mathbf{D}g_*\mathbf{D}g^*\Psi_K(\xi)[d]] \circ [j_K^*\beta(\Psi_K(N)[d])] \circ \alpha_K(N).
\end{aligned}
$$

We also know that $\beta$ is a morphism of functors, so

$$
\begin{array}{ccc}
\Psi_K(N) & \xrightarrow{\quad \beta(\Psi_K(N)[d]) \quad} & \mathbf{D}g_*\mathbf{D}g^*\Psi_K(N)[d] \\
\Big\downarrow{\scriptstyle \Psi_K(\xi)} & & \Big\downarrow{\scriptstyle \mathbf{D}g_*\mathbf{D}g^*\Psi_K(\xi)[d]} \\
\Psi_K\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d] & \xrightarrow[\beta\left(\Psi_K\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)]{} & \mathbf{D}g_*\mathbf{D}g^*\Psi_K\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}
\end{array}
$$

commutes and

$$
\begin{aligned}
G(\xi) = {}& [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ [j_K^*\mathbf{D}g_*\Psi_H\left(\delta\left(j_H^*\mathcal{M}\right)\right)] \circ \\
&\left[j_K^*\mathbf{D}g_*\gamma_K^H\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ \left[j_K^*\beta\left(\Psi_K\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ [j_K^*\Psi_K(\xi)] \circ \alpha_K(N).
\end{aligned}
$$

Finally, since $\alpha_K$ is a morphism of functors, we get the following commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\alpha_K(N)} & j_K^*\Psi_K(N) \\ \Big\downarrow{\scriptstyle \xi} & & \Big\downarrow{\scriptstyle j_K^*\Psi_K(\xi)} \\ \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d] & \xrightarrow[\alpha_K\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right)]{} & j_K^*\Psi_K \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d] \end{array}$$

and we obtain

$$\begin{aligned} G(\xi) = {} & [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ [j_K^*\mathbf{D}g_*\Psi_H\left(\delta\left(j_H^*\mathcal{M}\right)\right)] \circ \\ & \left[j_K^*\mathbf{D}g_*\gamma_K^H\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ \left[j_K^*\beta\left(\Psi_K \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ \\ & \left[\alpha_K\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right)\right] \circ \xi. \end{aligned}$$

Finally, since the morphism (*) in the definition of $G$ is an isomorphism when $N$ is finitely generated over $\mathcal{U}(\mathfrak{g})$, we get that the composition

$$\mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\,K)}(N,\,\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]) \overset{G}{\to} \mathrm{Hom}_{\mathbf{D}^b(\mathfrak{g},\,K)}(N,\,j_K^*\mathbf{D}g_*\mathcal{M})$$

is an isomorphism when $N$ is finitely generated over $\mathcal{U}(\mathfrak{g})$. This proves that

$$\begin{aligned} & [j_K^*\mathbf{D}g_*\left(\epsilon_H(\mathcal{M})\right)] \circ [j_K^*\mathbf{D}g_*\Psi_H\left(\delta\left(j_H^*\mathcal{M}\right)\right)] \circ \\ & \left[j_K^*\mathbf{D}g_*\gamma_K^H\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ \left[j_K^*\beta\left(\Psi_K \mathrm{R}\Gamma_H^K j_H^*\mathcal{M}\right)\right] \circ \\ & \left[\alpha_K\left(\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]\right)\right] \end{aligned}$$

is an isomorphism from $\mathrm{R}\Gamma_H^K j_H^*\mathcal{M}[-d]$ to $j_K^*\mathbf{D}g_*\mathcal{M}$. □

### 3.3. Bott-Borel-Weil theorem for Zuckerman functor

Suppose that $G$ is a semisimple and simply connected complex algebraic group and denote by $\mathfrak{g}$ its associated semisimple Lie algebra.

The notations of chapter 1 will be used but will not be recalled here.

Let $I \subset I_0$ and $X_I = G/Q_I$ be the generalized flag manifold associated to $I$. In the sequel, we will denote by $d_I$ the complex dimension of $X_I$ previously denoted by $d_{X_I}$. Set $j_I\ :\ \{\mathrm{pt}\} \hookrightarrow G/Q_I$ the morphism associated to $eQ_I$. By the equivalence stated in proposition 2.2.21, we know that the functor

$$j_I^*\ :\ \mathbf{D}^b(\mathcal{D}_{X_I},\,G) \overset{\sim}{\to} \mathbf{D}^b(\mathfrak{g},\,Q_I)$$

is an equivalence of categories. Let

$$\Psi_I\ :\ \mathbf{D}^b(\mathfrak{g},\,Q_I) \overset{\sim}{\to} \mathbf{D}^b(\mathcal{D}_{X_I},\,G)$$

be its quasi-inverse defined p.57. If $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$, we denote by $\mathcal{O}_{X_I}(\lambda)$ the $G$-equivariant $\mathcal{O}_{X_I}$-module corresponding to $V_I(\lambda)$ under the equivalence of categories on the $\mathcal{O}$-modules level, described in the proposition 2.1.17. We see that if $M_I(\lambda)$ is the generalized Verma module associated to $I$ (see p.18), then the quasi-$G$-equivariant $\mathcal{D}_{X_I}$-module corresponding to the $(\mathfrak{g}, Q_I)$-module $M_I(\lambda)$ is isomorphic to

$$\mathcal{D}\mathcal{O}_{X_I}(\lambda) = \mathcal{D}_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda).$$

Now let us recall the following relative version of the Bott-Borel-Weil theorem.

PROPOSITION 3.3.1. *Let $I \subset J \subset I_0$ and denote by $g : X_I \to X_J$ the canonical projection. For $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$, we have the following.*

(1) *If there exists some $\alpha \in \Delta_J$ satisfying $(\lambda + \delta - 2\delta(\mathfrak{u}_I), \alpha) = 0$, then $Rg_*\mathcal{O}_{X_I}(\lambda) = 0$.*
(2) *Otherwise, take $w$ as the unique element of $W_J$ such that*

$$(w(\lambda + \delta - 2\delta(\mathfrak{u}_I), \alpha) > 0$$

*for every $\alpha \in \Delta^+_J$. Then we have*

$$Rg_*\mathcal{O}_{X_I}(\lambda) \cong \mathcal{O}_{X_J}(w \cdot (\lambda - 2\delta(\mathfrak{u}_I)) + 2\delta(\mathfrak{u}_J))[-(\ell(w_J w) - \ell(w_I))].$$

PROOF. See [**16**]. □

We can now establish a "Zuckerman version" of this result, where Zuckerman functor is substituted to the direct image functor.

THEOREM 3.3.2. *Let $I \subset J \subset I_0$ and denote by $g : X_I \to X_J$ the canonical projection. Let $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$.*

(1) *If $\lambda$ is affinely singular for $\mathfrak{q}_J$, then $\mathrm{R}\Gamma^{Q_J}_{Q_I} M_I(\lambda) = 0$.*
(2) *Otherwise, letting $w$ be the unique element of $W_J$ such that*

$$(w(\lambda + \delta), \alpha) > 0$$

*for every $\alpha \in \Delta^+_J$, we have*

$$\mathrm{R}\Gamma^{Q_J}_{Q_I} M_I(\lambda) \cong M_J(w \cdot \lambda)[\ell(w) - \#(\Delta_J \backslash \Delta_I)]$$

*as $(\mathfrak{g}, Q_J)$-modules.*

PROOF. Since

$$M_I(\lambda) \cong j^*_I \mathcal{D}\mathcal{O}_{X_I}(\lambda)$$

as $(\mathfrak{g}, Q_I)$-modules, the theorem 3.2.1 gives

$$\mathrm{R}\Gamma^{Q_J}_{Q_I} M_I(\lambda) \cong \mathrm{R}\Gamma^{Q_J}_{Q_I} j^*_I \mathcal{D}\mathcal{O}_{X_I}(\lambda) \cong j^*_J \mathbf{D} g_* \mathcal{D}\mathcal{O}_{X_I}(\lambda)[d_J - d_I].$$

Using corollary 1.3.7, we obtain

$$\begin{aligned} d_J - d_I &= \dim^{\mathbb{C}} X_J - \dim^{\mathbb{C}} X_I \\ &= \dim^{\mathbb{C}} \mathfrak{q}_I - \dim^{\mathbb{C}} \mathfrak{q}_J \\ &= \#\Delta_I - \#\Delta_J + \ell(w_J) - \ell(w_I) \\ &= \ell(w_J) - \ell(w_I) - \#(\Delta_J \backslash \Delta_I) \end{aligned}$$

since $\Delta_I \subset \Delta_J$ thus

$$\mathrm{R}\Gamma_{Q_I}^{Q_J} M_I(\lambda) \cong j_J^* \mathbf{D} g_* \mathcal{D}\mathcal{O}_{X_I}(\lambda)[\ell(w_J) - \ell(w_I) - \#(\Delta_J \backslash \Delta_I)]. \tag{17}$$

Since $\mathcal{D}\mathcal{O}_{X_I}(\lambda)$ is flat over $\mathcal{D}_{X_I}$, we have

$$\begin{aligned} \mathcal{D}_{X_J \leftarrow X_I} \otimes^L_{\mathcal{D}_{X_I}} \mathcal{D}\mathcal{O}_{X_I}(\lambda) &= \mathcal{D}_{X_J \leftarrow X_I} \otimes_{\mathcal{D}_{X_I}} \left( \mathcal{D}_{X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda) \right) \\ &\cong \mathcal{D}_{X_J \leftarrow X_I} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda) \\ &\cong \left( g^{-1}\mathcal{D}_{X_J} \otimes_{g^{-1}\mathcal{O}_{X_J}} \Omega_{X_I/X_J} \right) \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda) \\ &\cong g^{-1}\mathcal{D}_{X_J} \otimes_{g^{-1}\mathcal{O}_{X_J}} \left( \Omega_{X_I/X_J} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda) \right) \qquad (18) \end{aligned}$$

as $g^{-1}\mathcal{D}_{X_J}$-modules. By definition of $\Omega_{X_I/X_J}$, the character of the torus $T$ associated to $\mathfrak{h}$ on

$$\Omega_{X_I/X_J} = \Omega_{X_I} \otimes_{g^{-1}\mathcal{O}_{X_J}} g^{-1}\Omega_{X_J}^{\otimes -1}$$

is the difference between the character of $\Omega_{X_I}$ and the one of $\Omega_{X_J}$ since the pull back operation by $g$ does not alterate the action on the fiber. We know that $\Omega_{X_I}$ has fiber $\bigwedge^{d_I}(\mathfrak{g}/\mathfrak{q}_I)^*$ with $d_I = \dim^{\mathbb{C}} X_I$ and

$$(\mathfrak{g}/\mathfrak{q}_I)^* = \bigoplus_{\alpha \in \Delta^+ \backslash \Delta_I^+} \mathbb{C}\alpha.$$

Thus the character of $T$ on $\Omega_{X_I}$ is equal to $\sum_{\alpha \in \Delta^+ \backslash \Delta_I^+} \alpha$ and, by similar reasoning on $X_J$, we obtain that the character of $T$ on $\Omega_{X_I/X_J}$ is equal to

$$\gamma_{I,J} = \sum_{\alpha \in \Delta^+ \backslash \Delta_I^+} \alpha - \sum_{\alpha \in \Delta^+ \backslash \Delta_J^+} \alpha. \tag{19}$$

Since $I$ is included in $J$, we have $\Delta_I^+ \subset \Delta_J^+$ and then $\Delta^+ \backslash \Delta_J^+$ is a subset of $\Delta^+ \backslash \Delta_I^+$. This observation leads to the equality

$$\gamma_{I,J} = \sum_{\alpha \in (\Delta^+ \backslash \Delta_I^+) \backslash (\Delta^+ \backslash \Delta_J^+)} \alpha.$$

Checking that

$$(\Delta^+ \backslash \Delta_I^+) \backslash (\Delta^+ \backslash \Delta_J^+) = (\Delta^+ \backslash \Delta_I^+) \cap \Delta_J^+ = (\Delta^+ \cap \Delta_J^+) \backslash \Delta_I^+ = \Delta_J^+ \backslash \Delta_I^+,$$

we obtain that

$$\gamma_{I,J} = \sum_{\alpha\in\Delta_J^+\setminus\Delta_I^+} \alpha.$$

We deduce that, as $g^{-1}\mathcal{O}_{X_J}$-modules, we have

$$\Omega_{X_I/X_J} \cong \mathcal{O}_{X_I}(\gamma_{I,J})$$

and

$$\Omega_{X_I/X_J} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda) \cong \mathcal{O}_{X_I}(\lambda+\gamma_{I,J}).$$

Consequently, the equality (18) becomes

$$\mathcal{D}_{X_J\leftarrow X_I} \otimes^L_{\mathcal{D}_{X_I}} \mathcal{D}\mathcal{O}_{X_I}(\lambda) \cong g^{-1}\mathcal{D}_{X_J} \otimes_{g^{-1}\mathcal{O}_{X_J}} \mathcal{O}_{X_I}(\lambda+\gamma_{I,J})$$

and the projection formula (in the algebraic setting) gives

$$\begin{aligned} \mathbf{D}g_*\mathcal{D}\mathcal{O}_{X_I}(\lambda) &\cong Rg_*\left(g^{-1}\mathcal{D}_{X_J} \otimes_{g^{-1}\mathcal{O}_{X_J}} \mathcal{O}_{X_I}(\lambda+\gamma_{I,J})\right) \\ &\cong \mathcal{D}_{X_J} \otimes_{\mathcal{O}_{X_J}} Rg_*\mathcal{O}_{X_I}(\lambda+\gamma_{I,J}). \end{aligned} \tag{20}$$

Now we want to apply the Bott-Borel-Weil theorem 3.3.1. Using the notation (5) introduced p.17, the equality (19) can be written as $\gamma_{I,J} = 2\delta(\mathfrak{u}_I) - 2\delta(\mathfrak{u}_J)$. Thus we obtain thanks to (6) that

$$\begin{aligned} (\lambda+\gamma_{I,J}+\delta-2\delta(\mathfrak{u}_I),\, \alpha) &= (\lambda+2\delta(\mathfrak{u}_I)-2\delta(\mathfrak{u}_J)+\delta-2\delta(\mathfrak{u}_I),\, \alpha) \\ &= (\lambda-2\delta(\mathfrak{u}_J)+\delta,\, \alpha) \\ &= (\lambda+\delta,\, \alpha) - 2(\delta(\mathfrak{u}_J),\, \alpha) \\ &= (\lambda+\delta,\, \alpha) \end{aligned}$$

for all $\alpha \in \Delta_J$. Moreover, since $\delta(\mathfrak{u}_J)$ is invariant under the action of the Weyl group $W_J$, we get

$$w(\lambda+\gamma_{I,J}+\delta-2\delta(\mathfrak{u}_I)) = w(\lambda-2\delta(\mathfrak{u}_J)+\delta) = w(\lambda+\delta)-2\delta(\mathfrak{u}_J)$$

and thus

$$(w(\lambda+\gamma_{I,J}+\delta-2\delta(\mathfrak{u}_I)),\, \alpha) = (w(\lambda+\delta),\, \alpha)$$

for all $w \in W_J$ and $\alpha \in \Delta_J$. Consequently, the proposition 3.3.1 gives

$$Rg_*\mathcal{O}_{X_I}(\lambda+\gamma_{I,J}) = \begin{cases} 0 & \text{in case (1)} \\ \mathcal{O}_{X_J}(w\cdot\lambda)[-(\ell(w_Jw)-\ell(w_I))] & \text{in case (2)} \end{cases}$$

and equation (20) implies

$$\mathbf{D}g_*\mathcal{D}\mathcal{O}_{X_I}(\lambda) = \begin{cases} 0 & \text{in case (1)} \\ \mathcal{D}\mathcal{O}_{X_J}(w\cdot\lambda)[-(\ell(w_Jw)-\ell(w_I))] & \text{in case (2).} \end{cases}$$

Since $\ell(w_Jw) = \ell(w_J) - \ell(w)$ and $I \subset J$, we have

$$-(\ell(w_Jw)-\ell(w_I)) + \ell(w_J) - \ell(w_I) = \ell(w)$$

and this concludes the proof. □

### 3.4. Generalized Penrose transform and its algebraic variant

Now consider the following correspondence of generalized flag manifolds :

(21)
$$\begin{array}{ccc} & X_{I\cap J} & \\ {}^{f}\swarrow & & \searrow^{g} \\ X_I & & X_J \end{array}$$

where $I$, $J$ are subsets of $I_0$ and $f$ and $g$ are the canonical projections.

Since $f$ and $g$ are morphisms of $G$-manifolds, the functor

$$\mathcal{R}_I^J = \mathbf{D}g_*\mathbf{D}f^* \ : \ \mathbf{D}^b \operatorname{Mod}(\mathcal{D}_{X_I}) \to \mathbf{D}^b \operatorname{Mod}(\mathcal{D}_{X_J})$$

induces a functor

$$\mathcal{R}_I^J = \mathbf{D}g_*\mathbf{D}f^* \ : \ \mathbf{D}^b \operatorname{Mod}(\mathcal{D}_{X_I},\, G) \to \mathbf{D}^b \operatorname{Mod}(\mathcal{D}_{X_J},\, G)$$

which is called the *quasi-$G$-equivariant generalized Penrose transform* associated to $I \subset J$.

Theorem 3.4.1. *The functor*

$$\mathbb{R}_I^J \ : \ \mathbf{D}^b \operatorname{Mod}(\mathfrak{g},\, Q_I) \to \mathbf{D}^b \operatorname{Mod}(\mathfrak{g},\, Q_J)$$

*which corresponds to the Penrose transformation $\mathcal{R}_I^J$ under the correspondance of categories given in proposition 2.2.18 is*

$$\mathbb{R}_I^J = \mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} \mathcal{F}_{Q_I}^{Q_{I\cap J}}[d_{I\cap J} - d_J].$$

Proof. We have successively

$$\begin{aligned} j_J^*\mathbf{D}g_*\mathbf{D}f^*\Psi_I &\cong \mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} j_{I\cap J}^*\mathbf{D}f^*\Psi_I[d_{I\cap J} - d_J] \\ &\cong \mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} \mathcal{F}_{Q_I}^{Q_{I\cap J}} j_I^*\Psi_I[d_{I\cap J} - d_J] \\ &\cong \mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} \mathcal{F}_{Q_I}^{Q_{I\cap J}}[d_{I\cap J} - d_J] \end{aligned}$$

in $\mathbf{D}^b \operatorname{Mod}(\mathfrak{g},\, Q_I)$. □

Now we explain a method to analyze $\mathbb{R}_I^J M_I(\lambda)$ for $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$ inspired from [**64**]. Set

$$\Gamma = \{x \in W_I \ : \ x^{-1}\Delta_{I\cap J}^+ \subset \Delta_I^+\}.$$

It is well known that

$$\begin{aligned} \Gamma &= \{x \in W_I \ : \ x^{-1}\Delta_{I\cap J}^+ \subset \Delta_I^+\} \\ &= \{x \in W_I \ : \ x \text{ is the shortest element of } W_{I\cap J}x\} \\ &= \{x \in W_I \ : \ (w(\lambda+\delta),\, \alpha) > 0 \text{ for any } \alpha \in \Delta_{I\cap J}^+\} \end{aligned}$$

If $x \in \Gamma$ we have

$$\begin{aligned} (x\cdot\lambda,\, \alpha^\vee) &= (x(\lambda+\delta),\, \alpha^\vee) - (\delta,\, \alpha^\vee) \\ &= (x(\lambda+\delta),\, \alpha^\vee) - 1 \geq 0 \end{aligned}$$

for every simple root $\alpha$ belonging to $\Delta^+_{I\cap J}$. Thus $x\cdot\lambda$ is an element of $(\mathfrak{h}^*_{\mathbb{Z}})_{I\cap J}$ for every $x\in\Gamma$.

By Lepowsky [**60**] and Rocha-Caridi [**72**], we have the following resolution of the finite dimensional $\mathfrak{l}_I$-module $V_I(\lambda)$:

$$0\to N_n\to N_{n-1}\to\cdots\to N_1\to N_0\to V_I(\lambda)\to 0 \tag{22}$$

with $n=\dim^{\mathbb{C}}(\mathfrak{l}_I/\mathfrak{l}_I\cap\mathfrak{q}_J)$ and

$$N_k=\bigoplus_{\substack{x\in\Gamma\\ \ell(x)=k}}\mathcal{U}(\mathfrak{l}_I)\otimes_{\mathcal{U}(\mathfrak{l}_I\cap\mathfrak{p}_J)}V_{I\cap J}(x\cdot\lambda).$$

Letting $u_{I\cap J}$ acting trivially on $V_{I\cap J}(x\cdot\lambda)$, the Poincaré-Birkhoff-Witt theorem gives that

$$\mathcal{U}(\mathfrak{l}_I)\otimes_{\mathcal{U}(\mathfrak{l}_I\cap\mathfrak{p}_J)}V_{I\cap J}(x\cdot\lambda)\cong\mathcal{U}(\mathfrak{q}_I)\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda)$$

as $\mathcal{U}(\mathfrak{l}_I)$-modules. Moreover, the action of $\mathfrak{u}_I$ is trivial on

$$\mathcal{U}(\mathfrak{q}_I)\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda).$$

This can be verified using the fact $\mathfrak{u}_I\mathcal{U}(\mathfrak{q}_I)=\mathcal{U}(\mathfrak{q}_I)\mathfrak{u}_I$ since $[\mathfrak{q}_I,\,\mathfrak{u}_I]\subset\mathfrak{u}_I$ and then

$$\begin{aligned}\mathfrak{u}_I(\mathcal{U}(\mathfrak{q}_I)\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda)) &= (\mathfrak{u}_I\mathcal{U}(\mathfrak{q}_I))\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda))\\ &= \mathcal{U}(\mathfrak{q}_I)\mathfrak{u}_I\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda))\\ &= \mathcal{U}(\mathfrak{q}_I)\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}\mathfrak{u}_IV_{I\cap J}(x\cdot\lambda))\\ &= \{0\}.\end{aligned}$$

Thus we obtain the following resolution of the $\mathfrak{q}_I$-module $V_I(\lambda)$:

$$0\to N'_n\to N'_{n-1}\to\cdots\to N'_1\to N'_0\to V_I(\lambda)\to 0 \tag{23}$$

with

$$N'_k=\bigoplus_{\substack{x\in\Gamma\\ \ell(x)=k}}\mathcal{U}(\mathfrak{q}_I)\otimes_{\mathcal{U}(\mathfrak{q}_{I\cap J})}V_{I\cap J}(x\cdot\lambda).$$

The application of the functor $\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{q}_I)}-$ to resolution (23), we obtain the following resolution of the $(\mathfrak{g},\,Q_{I\cap J})$-module $\mathcal{F}^{Q_{I\cap J}}_{Q_I}M_I(\lambda)$:

$$0\to\widetilde{N}_n\to\widetilde{N}_{n-1}\to\cdots\to\widetilde{N}_1\to\widetilde{N}'_0\to\mathcal{F}^{Q_{I\cap J}}_{Q_I}M_I(\lambda)\to 0 \tag{24}$$

with

$$\widetilde{N}_k=\bigoplus_{\substack{x\in\Gamma\\ \ell(x)=k}}M_{I\cap J}(x\cdot\lambda).$$

Define the following subset of $\Gamma$:

$$\Gamma(\lambda)=\{\nu\in\Gamma\ :\ (\nu(\lambda+\delta),\,\alpha)\neq 0\text{ for all }\alpha\in\Delta_J\}.$$

COROLLARY 3.4.2. *Let $x \in \Gamma$. We have*

$$\mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} M_{I\cap J}(x \cdot \lambda) \neq 0$$

*if and only if $x \in \Gamma(\lambda)$ and in such case, we have*

$$\mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} M_{I\cap J}(x \cdot \lambda) = M_J\left((y_x x) \cdot \lambda)\right)[\ell(y_x) - \#(\Delta_J \backslash \Delta_{I\cap J})]$$

*where $y_x$ is the unique element of $W_J$ such that*

$$(y_x x(\lambda + \delta),\, \alpha) > 0.$$

PROOF. If there is an element $\alpha \in \Delta_J$ such that

$$0 = (x(\lambda + \delta),\, \alpha) = (x \cdot \lambda + \delta,\, \alpha),$$

then theorem 3.3.2 gives that $\mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} M_{I\cap J}(x\cdot\lambda) = 0$. And for every $x \in \Gamma(\lambda)$, theorem 3.3.2 gives

$$\mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} M_{I\cap J}(x \cdot \lambda) = M_J\left(y_x \cdot (x \cdot \lambda)\right)[\ell(y_x) - \#(\Delta_J \backslash \Delta_{I\cap J})]$$

where $y_x$ is the unique element of $W_J$ such that

$$(y_x(x \cdot \lambda + \delta),\, \alpha) > 0$$

for every $\alpha \in \Delta_J^+$. The proof is complete since

$$y_x \cdot (x \cdot \lambda) = (y_x x) \cdot \lambda$$

and

$$(y_x(x \cdot \lambda + \delta),\, \alpha) = (y_x x(\lambda + \delta),\, \alpha)\,. \qquad \square$$

Hence we obtain

$$\mathrm{R}\Gamma_{Q_{I\cap J}}^{Q_J} \widetilde{N}_k = \bigoplus_{\substack{x\in\Gamma(\lambda)\\ \ell(x)=k}} M_J\left((y_x x) \cdot \lambda\right)[\ell(y_x) - \#(\Delta_J \backslash \Delta_{I\cap J})] \tag{25}$$

where $y_x$ is the unique element of $W_J$ such that $(y_x x(\lambda + \delta),\, \alpha) > 0$ for all $\alpha \in \Delta_J^+$.

THEOREM 3.4.3. *For any $\lambda \in (\mathfrak{h}_{\mathbb{Z}}^*)_I$, there exists a family*

$$\{M(k)^\bullet \;:\; k \in \mathbb{N}\}$$

*of objects of $\mathbf{D}^b\,\mathrm{Mod}(\mathfrak{g},\, Q_J)$ satisfying the following conditions:*

(1) $M(0)^\bullet = \mathbb{R}_I^J M_I(\lambda)$,
(2) $M(k)^\bullet = 0$ *for* $k > \dim^{\mathbb{C}}(\mathfrak{l}_I/\mathfrak{l}_I \cap \mathfrak{q}_J)$,
(3) *we have a distinguished triangle* $\mathcal{C}(k)^\bullet \to M(k)^\bullet \to M(k+1)^\bullet \xrightarrow{+1}$ *where*

$$\mathcal{C}(k)^\bullet = \bigoplus_{\substack{x\in\Gamma(\lambda)\\ \ell(x)=k}} M_J\left((y_x x) \cdot \lambda\right)[\ell(y_x x) - \ell(w_J w_{I\cap J})].$$

PROOF. In order to avoid awkward notations in the proof, set $n = \dim^{\mathbb{C}}(\mathfrak{l}_I/\mathfrak{l}_I \cap \mathfrak{q}_J)$. for every integer $0 \leq k \leq n$, define

$$N(k)^\bullet = [\cdots \to 0 \to \widetilde{N}_n \to \widetilde{N}_{n-1} \to \cdots \to \widetilde{N}_k \to 0 \to \cdots]$$

where each $\widetilde{N}_j$ $(k \leq j \leq n)$ has degree $-j$. For $k > n$, define $N(k)^\bullet = 0$. Now define

$$M(k)^\bullet = \mathrm{R}\Gamma^{Q_J}_{Q_{I\cap J}} N(k)^\bullet [d_{I\cap J} - d_J]$$

for $k \in \mathbb{N}$. The statements (1) and (2) are obvious by construction and it remains to show (3). Applying the functor $\mathrm{R}\Gamma^{Q_J}_{Q_{I\cap J}}[d_{I\cap J} - d_J]$ to the distinguished triangle

$$\widetilde{N}_k[k] \to N(k)^\bullet \to N(k+1)^\bullet \overset{+1}{\to}$$

we obtain a distinguished triangle

$$\mathrm{R}\Gamma^{Q_J}_{Q_{I\cap J}} \widetilde{N}_k[k + d_{I\cap J} - d_J] \to M(k)^\bullet \to M(k+1)^\bullet \overset{+1}{\to} .$$

After application of the equality (25), it suffices to check that the shift is correct i.e. that

$$\ell(x) + \ell(y_x) + d_{I\cap J} - d_J - \#(\Delta_J \backslash \Delta_{I\cap J}) = \ell(y_x x) - \ell(w_J w_{I\cap J})$$

for every $x \in \Gamma(\lambda)$ such that $\ell(x) = k$. On one hand we know that since $I \cap J \subset J$, we have $W_{I\cap J} \subset W_J$ and $\Delta_{I\cap J} \subset \Delta_J$ hence $\ell(w_J w_{I\cap J}) = \ell(w_J) - \ell(w_{I\cap J})$ and $\#(\Delta_J \backslash \Delta_{I\cap J}) = \#\Delta_J - \#\Delta_{I\cap J}$. Moreover, corollary 1.3.7 gives

$$d_{I\cap J} - d_J = \#\Delta_J - \#\Delta_{I\cap J} - \ell(w_J) + \ell(w_{I\cap J})$$

hence

$$d_{I\cap J} - d_J - \#(\Delta_J \backslash \Delta_{I\cap J}) = -\ell(w_J w_{I\cap J}).$$

It remains to prove that $\ell(y_x x) = \ell(y_x) + \ell(x)$. Since $x \in W_I$, we know that $x^{-1}\Delta^+_{I\cap J}$ is a subset of $\Delta^+_I$ and then

$$x\Delta^+ \cap \Delta^- = x\Delta^+_I \cap \Delta^-_I \subset \Delta^- \backslash \Delta_{I\cap J} \subset \Delta^- \backslash \Delta_J,$$

and elements of $\Delta^- \backslash \Delta_J$ are invariants under the action of $W_J$. Hence $y_x(x\Delta^+ \cap \Delta^-) \subset \Delta^- \backslash \Delta_J$ and $\#(y_x(x\Delta^+ \cap \Delta^-) \cap \Delta^+) = 0$. Since

$$y_x x\Delta^- \cap \Delta^+ = \big(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+\big) \uplus \big(y_x(x\Delta^- \cap \Delta^-) \cap \Delta^+\big)$$

where $\uplus$ means disjoint union, we have

$$\begin{aligned} \ell(y_x x) &= \#(y_x x\Delta^- \cap \Delta^+) \\ &= \#(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \#(y_x(x\Delta^- \cap \Delta^-) \cap \Delta^+) \\ &= \#(y_x(x\Delta^- \cap \Delta^+) \cap \Delta^+) + \#(y_x\Delta^- \cap \Delta^+) \\ &\quad -\#(y_x(x\Delta^+ \cap \Delta^-) \cap \Delta^+) \\ &= \ell(x) + \ell(y_x). \quad \square \end{aligned}$$

The latter result can be stated using notations from [**64**]. If $\lambda \in (\mathfrak{h}^*_{\mathbb{Z}})_I$ and $x \in \Gamma(\lambda)$, set

$$m(x) = \ell(w_J y_x) - \ell(w_{I\cap J})$$

where $y_x$ is the unique element of $W_J$ such that $(y_x x(\lambda+\delta), \alpha) > 0$ for all $\alpha \in \Delta_J^+$. We have (see [**64**])

$$m(x) = \#\{\alpha \in \Delta_J^+ \backslash \Delta_I \ : \ (x(\lambda+\delta), \alpha) > 0\}.$$

Since $y_x \in W_J$, the shift obtained in the definition of $\mathcal{C}(k)^\bullet$ in theorem 3.4.3 is equal to

$$\begin{aligned} \ell(y_x x) - \ell(w_J w_{I\cap J}) &= \ell(y_x) + \ell(x) - \ell(w_J) + \ell(w_{I\cap J}) \\ &= \ell(x) - \ell(w_J y_x) + \ell(w_{I\cap J}) \\ &= \ell(x) - m(x). \end{aligned}$$

Hence the shift obtained here corresponds exactly with the shift obtained in [**64**].

# Nomenclature

## General notations

| | |
|---|---|
| $\#A$ | number of elements of the finite set $A$ |
| $A\backslash B$ | complementary set to $B$ in $A$ |
| $\{\mathrm{pt}\}$ | the set consisting of a single element |
| $\mathbb{N}$ | set of non-negative integers |
| $\mathbb{N}_0$ | set of positive integers, $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$. |
| $\mathbb{Z}$ | ring of integers |
| $\mathbb{R}$ | field of real numbers |
| $\mathbb{C}$ | field of complex numbers |
| $X \times_S Y$ | fiber product over $S$ |
| $\dim^{\mathbb{C}} V$ | complex dimension of $V$ |

## Lie algebras and Lie groups, representation theory

| | |
|---|---|
| $[\cdot,\cdot]$ | Lie bracket, p2 |
| $(\cdot,\cdot)$ | p6 |
| $\prec, \succ$ | total ordering on $\mathfrak{h}^*$, p5 |
| ad | adjoint representation, p2 |
| $\alpha^\vee$ | coroot associated to $\alpha$, p6 |
| $\mathfrak{b}^\pm$ | standard Borel subalgebras, p15 |
| $\beta$ | Killing form, p5 |
| $\delta$ | half sum of the positive roots of $\mathfrak{g}$, p8 |
| $\delta_I$ | half sum of the positive roots of $\mathfrak{q}_I$, p17 |
| $\delta(\mathfrak{u}_I)$ | p17 |
| $\Delta$ | set of roots, p4 |
| $\Delta^+$ | set of positive roots, p5 |
| $\Delta^-$ | set of negative roots, p5 |
| $\Delta_I$ | p16 |
| $\Delta_I^\pm$ | p16 |
| $D_i\mathfrak{g}$ | lower central series of $\mathfrak{g}$, p3 |
| exp | exponential map, p9 |
| $\mathcal{F}_K^H$ | forgetful functor, p60 |
| $\mathfrak{g}$ | complex Lie algebra, p2 |

| | |
|---|---|
| $\mathfrak{g}_\alpha$ | root space associated to $\alpha$, p4 |
| $\mathfrak{g}^{(i)}$ | derived series of $\mathfrak{g}$, p2 |
| $G(\mathfrak{g})$ | simply connected Lie group of $\mathfrak{g}$, p11 |
| $\Gamma$ | p66 |
| $\Gamma_H^K$ | Zuckerman functor, p60 |
| $\Gamma(\mathfrak{q})$ | p18 |
| $\Gamma(\lambda)$ | p67 |
| $\mathfrak{h}$ | Cartan subalgebra, p4 |
| $\mathfrak{h}^*$ | dual of $\mathfrak{h}$, p4 |
| $\mathfrak{h}_\mathbb{R}$ | subset of $\mathfrak{h}$, p6 |
| $\mathfrak{h}_\mathbb{Z}^*$ | integral elements for $\mathfrak{g}$, p13 |
| $(\mathfrak{h}_\mathbb{Z}^*)_I$ | elements integral for $\mathfrak{g}$ and dominant for $\mathfrak{q}_I$, p21 |
| $(\mathfrak{h}_\mathbb{Z}^*)_{I_0}$ | integral and dominant elements for $\mathfrak{g}$, p14 |
| $\mathrm{Hom}_\mathfrak{g}(-,-)$ | $\mathfrak{g}$-modules homomorphisms, p11 |
| $\mathrm{Hom}_{(\mathfrak{g},H)}(-,-)$ | $(\mathfrak{g}, H)$-modules homomorphisms, p57 |
| $I_\mathfrak{q}$ | p18 |
| $\ell$ | length function on the Weyl group, p9 |
| $\mathfrak{l}_I$ | reductive part of $\mathfrak{q}_I^\pm$, p16 |
| $\lambda_i$ | p6 |
| $\mathscr{L}ie$ | Lie functor, p9 |
| $m(x)$ | p70 |
| $M_{I_0}(\lambda)$ | Verma module associated to $\lambda$, p15 |
| $M_I(\lambda)$ | generalized Verma module, p18 |
| $\mathfrak{n}^\pm$ | nilpotent subalgebra, p15 |
| $\mathfrak{q}_I^\pm$ | standard parabolic subalgebra associated to $I$, p16 |
| $\mathrm{rad}\,\mathfrak{g}$ | radical of $\mathfrak{g}$, p2 |
| $\rho$ | representation morphism, p11 |
| $\mathbb{R}_I^J$ | algebraic Penrose transform, p66 |
| $\mathcal{S}$ | simple system of roots, p5 |
| $\sigma_\alpha$ | reflection associated to $\alpha$, p7 |
| $\mathfrak{u}_I^\pm$ | nilpotent part of $\mathfrak{q}_I^\pm$, p16 |
| $\mathcal{U}(\mathfrak{g})$ | universal enveloping algebra, p13 |
| $V_\lambda$ | weight space associated to $\lambda$, p12 |
| $V_\mathfrak{g}(\lambda)$ | irreducible representation of $\mathfrak{g}$ associated to $\lambda$, p14 |
| $V_{Q_I}(\lambda)$ | irred. represent. of $Q_I$ with highest weight $\lambda$, p21 |
| $V_I(\lambda)$ | irred. represent. of $Q_I$ with highest weight $\lambda$, p21 |
| $w \cdot \lambda$ | affine action of the Weyl group, p8 |
| $W_\alpha$ | $\alpha$-wall, p7 |
| $W_\mathfrak{g}$ | Weyl group of $\mathfrak{g}$, p7 |
| $w_\mathfrak{g}$ | longest element of $W_\mathfrak{g}$, p9 |
| $W_I$ | p17 |
| $w_I$ | longest element of $W_I$, p17 |
| $y_x$ | p68 |

| | |
|---|---|
| $\mathfrak{z}(\mathfrak{g})$ | center of $\mathfrak{g}$, p4 |

## Categories

| | |
|---|---|
| **LieA** | category of complex Lie algebras, p9 |
| **LieG** | category of complex Lie groups, p9 |
| $\mathrm{Mod}(\mathfrak{g})$ | category of $\mathfrak{g}$-modules, p12 |
| $\mathrm{Mod}(\mathfrak{g}, H)$ | category of $(\mathfrak{g}, H)$-modules, p57 |
| $\mathrm{Mod}(\mathcal{O}_Z)$ | category of $\mathcal{O}_Z$-modules, p40 |
| $\mathrm{Mod}_G(\mathcal{O}_Z)$ | category of $G$-equivariant quasi-coherent $\mathcal{O}_Z$-modules, p40 |
| $\mathrm{Mod}_{G,\,\mathrm{coh}}(\mathcal{O}_Z)$ | full subcategory of $\mathrm{Mod}_G(\mathcal{O}_Z)$ consisting of coherent objects, p40 |
| $\mathrm{Mod}(\mathcal{D}_Z, G)$ | category of quasi-coherent, quasi-$G$-equivariant $\mathcal{D}_Z$-modules, p49 |
| $\mathrm{Mod}_G(\mathcal{D}_Z)$ | full subcategory of $\mathrm{Mod}(\mathcal{D}_Z, G)$ of $G$-equivariant $\mathcal{D}_Z$-modules, p49 |
| $\mathrm{Mod}_{\mathrm{lf}}(\mathcal{D}_Z, G)$ | p50 |
| $\mathbf{D}^b(\mathfrak{g}, H)$ | derived category built from $\mathrm{Mod}(\mathfrak{g}, H)$, p57 |
| $\mathbf{D}^b(\mathcal{D}_Z, G)$ | derived category of $\mathrm{Mod}(\mathcal{D}_Z, G)$, p50 |
| $\mathbf{D}^b_{\mathrm{coh}}(\mathcal{D}_Z, G)$ | derived category of $\mathrm{Mod}_{\mathrm{coh}}(\mathcal{D}_Z, G)$, p50 |

## Differential geometry

| | |
|---|---|
| $d_{X_I}$ | complex dimension of $X_I$, p62 |
| $\partial, \overline{\partial}$ | p25 |
| $G^\circ$ | connected component of $G$ at identity, p21 |
| $G \times_B V$ | Fibered product, p35 |
| $h$ | Hermitian metric, p26 |
| $\mathcal{H}^n_\alpha$ | Hopf manifold, p33 |
| $J$ | almost complex structure, p26 |
| $\Omega_h$ | fundamental form associated to $h$, p26 |
| $X_I$ | generalized flag manifold associated to $I$, p62 |

## $\mathcal{O}$-modules and $\mathcal{D}$-modules

| | |
|---|---|
| $-\otimes^D -$ | $\mathcal{D}$-module tensor product, p51 |
| $-\boxtimes-$ | $\mathcal{D}$-module exterior tensor product, p51 |
| $-\otimes-$ | $\mathcal{D}$-module tensor product, p51 |
| $\mathcal{D}_{X\to Y}$ | transfer bimodule, p53 |
| $\mathcal{D}_{Y\leftarrow X}$ | transfer bimodule, p53 |
| $\mathbf{D}f^*$ | $\mathcal{D}$-module inverse image functor, p54 |
| $\mathbf{D}f_*$ | $\mathcal{D}$-module direct image functor, p54 |

| | |
|---|---|
| $\mathcal{DO}_{X_I}(\lambda)$ | p63 |
| $\gamma_K^H$ | p60 |
| $\mathrm{Hom}_{\mathcal{D}_Z}(-,-)$ | p52 |
| $i$ | canonical embedding, p47 |
| $j_H$ | p48 |
| $\mu$ | action morphism, p38 |
| $\mathcal{O}_G \boxtimes \mathcal{D}_Z$ | p48 |
| $\mathcal{O}_{G/B}(V)$ | p35 |
| $\mathcal{O}_{G/B}(\lambda)$ | p35 |
| $\mathcal{O}_{G/Q}(\lambda)$ | p36 |
| $\Omega_{X/Y}$ | p53 |
| $q_1, q_2, q_3$ | p39 |
| $\pi$ | canonical projection, p47 |
| pr | canonical projection, p38 |
| $\Psi_H$ | p47 |
| $\mathrm{RHom}_{\mathcal{D}_Z}(-,-)$ | p53 |
| $\mathcal{R}_I^J$ | quasi-equivariant generalized Penrose transform, p66 |

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